Lecture 1: Introduction

André Martins

Deep Structured Learning Course, Fall 2019
Course Website

https://andre-martins.github.io/pages/
deepe-structured-learning-ist-fall-2019.html

There I’ll post:

• Syllabus
• Lecture slides
• Literature pointers
• Homework assignments
• ...
Outline

1 Introduction

2 Class Administrativia

3 Recap

   Linear Algebra

   Probability Theory

   Optimization
What is “‘Deep Learning’”? 

- Neural networks?
- Neural networks with many hidden layers?
- Anything beyond shallow (linear) models for statistical learning?
- Anything that learns representations?
- A form of learning that is really intense and profound?
What is “Deep Learning”?

- Neural networks?
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- Anything that learns representations?
- A form of learning that is really intense and profound?
Where is the “Structure”? 

- In the **input** objects (text, graphs, images, ...)
- In the **outputs** we want to predict (parsing, graph labeling, image segmentation, ...)
- In our **model** (convolutional networks, attention mechanisms)
- Related: **latent structure** (typically a way of encoding prior knowledge into the model)
Why Did Deep Learning Become Mainstream?

Lots of recent breakthroughs:

- Object recognition
- Speech and language processing
- Chatbots and dialog systems
- Self-driving cars
- Machine translation
- Solving games (Atari, Go)

No signs of slowing down…
Microsoft's new breakthrough: AI that's as good as humans at listening... on the phone

Microsoft's new speech-recognition record means professional transcribers could be among the first to lose their jobs to artificial intelligence.

By Liam Tung | October 19, 2016 -- 10:10 GMT (11:10 BST) | Topic: Innovation
A closer look at Google Duplex

Google's appointment booking AI wowed the crowd and raised concern at I/O.
Who is wearing glasses?

- man
- woman

Where is the child sitting?

- fridge
- arms

Is the umbrella upside down?

- yes
- no

How many children are in the bed?

- 2
- 1
Google unleashes deep learning tech on language with Neural Machine Translation

Systran launches neural machine translation engine in 30 languages

Language barriers represent one of the biggest challenges to develop business strategies among global markets. Now, thanks to advances in artificial intelligence and machine translation, these barriers are being broken down.
Siri and Alexa Are Fighting to Be Your Hotel Butler

By Hui-yong Yu and Spencer Soper
March 22, 2017, 9:00 AM GMT  Updated on March 22, 2017, 2:13 PM GMT

- Hotels are new frontier for voice-command technologies
- Wynn Las Vegas was first to install Alexa devices in December
AlphaGo Beats Go Human Champ: Godfather Of Deep Learning Tells Us Do Not Be Afraid Of AI

21 March 2016, 10:16 am EDT  By Aaron Mamit Tech Times

Last week, Google's artificial intelligence program AlphaGo dominated its match with South Korean world Go champion Lee Sedol, winning with a 4-1 score.

The achievement stunned artificial intelligence experts, who previously thought that Google's computer program would need at least 10 more years before developing enough to be able to beat a human world champion.
Why does deep learning work now, but not 20 years ago? Many of the core ideas were there, after all. But now we have:

- more data
- more computing power
- better software engineering
- a few algorithmic innovations (many layers, ReLUs, better initialization and learning rates, dropout, LSTMs, convolutional nets)
Why does gradient-based optimization work at all in neural nets despite the non-convexity?

One possible, partial answer:

- there are generally many hidden units
- there are many ways a neural net can approximately implement the desired input-output relationship
- we only need to find one
Main book:

Recommended Books

Secondary books:


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Outline

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- Linear Algebra
- Probability Theory
- Optimization
What This Class Is About

• Introduction to deep learning
• Introduction to structured prediction

• **Goal:** after finishing this class, you should be able to:
  • Understand how deep learning works without magic
  • Understand the intuition behind deep structured learning models
  • Apply the learned techniques on a practical problem (NLP, vision, ...)

• **Target audience:**
  • MSc/PhD students with basic background in ML and good programming skills
What This Class Is Not About

It’s **not** about:

- Just playing with a deep learning toolkit without learning the fundamental concepts
- Introduction to ML (see Mário Figueiredo’s *Statistical Learning* course and Jorge Marques’ *Estimation and Classification* course)
- Optimization (check João Xavier’s *Non-Linear Optimization* course)
- Natural Language Processing
- ...
Prerequisites

- Calculus and basic linear algebra
- Basic probability theory
- Basic knowledge of machine learning
- Programming (Python & PyTorch preferred)
- Helpful: basic optimization
Course Information

- **Instructors**: André Martins & Vlad Niculae
- **TAs**: Gonçalo Correia & Ben Peters
- **Location**: LT2 (North Tower, 4th floor)
- **Schedule**: Mondays/Fridays 10:00–11:30 (tentative)
- **Communication**: piazza.com/tecnico.ulisboa.pt/fall2019/pdeecdsl
Grading

• 4 homework assignments: 60%
  • Theoretical questions & implementation
  • Late days: 10% penalization each late day

• Final project (in groups of 2–3): 40%
  • Final class presentations & poster session (tentative)
• **Possible idea:** apply a deep learning technique to a structured problem relevant to your research (NLP, vision, robotics, ...)

• Otherwise, pick a project from a list of suggestions

• Must be finished this semester

• Four evaluation stages: project proposal (10%), midterm report (10%), final report (10%, conference paper format), class presentation (10%)

• List of project suggestions will be made available soon
Collaboration Policy

• Assignments are individual
• Students may discuss the questions, as long as they write their own answers and their own code
• If this happens, acknowledge with whom you collaborate!
• Zero tolerance on plagiarism!!
• Always credit your sources!!!
• This is the second year I’m teaching this class
• ... which means you’re the second batch of students taking it :) 
• Constructive feedback will be highly appreciated (and encouraged!)
Questions?

ALL QUESTIONS MUST BE SUBMITTED IN WRITING
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Quick Background Recap

Slide credits: Prof. Mário Figueiredo (taken from his LxMLS class)
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3. Recap

   Linear Algebra

   Probability Theory

   Optimization
Linear algebra provides (among many other things) a compact way of representing, studying, and solving linear systems of equations.
Linear Algebra

• Linear algebra provides (among many other things) a compact way of representing, studying, and solving linear systems of equations

• **Example**: the system

\[
\begin{align*}
4x_1 - 5x_2 &= -13 \\
-2x_1 + 3x_2 &= 9
\end{align*}
\]

can be written compactly as \(Ax = b\), where

\[
A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} -13 \\ 9 \end{bmatrix},
\]

and can be solved as

\[
x = A^{-1}b = \begin{bmatrix} 1.5 & 2.5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -13 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.
\]
Notation: Matrices and Vectors

- $A \in \mathbb{R}^{m \times n}$ is a matrix with $m$ rows and $n$ columns.

$$A = \begin{bmatrix}
A_{1,1} & \cdots & A_{1,n} \\
\vdots & \ddots & \vdots \\
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- $x \in \mathbb{R}^n$ is a vector with $n$ components,
  
  $x = \begin{bmatrix}
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    \vdots \\
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• A **(column) vector** is a matrix with \( n \) rows and 1 column.
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$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

- A (column) vector is a matrix with $n$ rows and 1 column.

- A matrix with 1 row and $n$ columns is called a row vector.
Matrix Transpose and Products

- Given matrix $A \in \mathbb{R}^{m \times n}$, its transpose $A^T$ is such that $(A^T)_{i,j} = A_{j,i}$. 

- A matrix $A$ is symmetric if $A^T = A$.

- Given matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, their product is $C = A B \in \mathbb{R}^{m \times p}$ where $C_{i,j} = \sum_{k=1}^{n} A_{i,k} B_{k,j}$.

- Inner product between vectors $x, y \in \mathbb{R}^n$: $\langle x, y \rangle = x^T y = y^T x = \sum_{i=1}^{n} x_i y_i \in \mathbb{R}$.

- Outer product between vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$: $x y^T \in \mathbb{R}^{n \times m}$, where $(x y^T)_{i,j} = x_i y_j$. 
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Properties of Matrix Products and Transposes

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- Matrix product is associative: \((AB)C = A(BC)\).

- In general, matrix product is not commutative: \(AB \neq BA\).

- Transpose of product: \((AB)^T = BA^T\).

- Transpose of sum: \((A + B)^T = A^T + B^T\).
Properties of Matrix Products and Transposes

- Given matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, their product is

$$C = A \cdot B \in \mathbb{R}^{m \times p} \text{ where } C_{i,j} = \sum_{k=1}^{n} A_{i,k} B_{k,j}$$

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Properties of Matrix Products and Transposes

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$$\|x\|_2 = \sqrt{\langle x, x \rangle} = \sqrt{x^Tx} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$
Norms

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• More generally, the $\ell_p$ norm of a vector $x \in \mathbb{R}^n$, where $p \geq 1$, is

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- Notable case: the \( \ell_{\infty} \) norm, \( \|x\|_{\infty} = \max\{|x_1|, ..., |x_n|\} \).
Norms

- The norm of a vector is (informally) its “magnitude.” Euclidean norm:

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- More generally, the $\ell_p$ norm of a vector $x \in \mathbb{R}^n$, where $p \geq 1$,

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- Notable case: the $\ell_1$ norm, $\|x\|_1 = \sum_i |x_i|.$

- Notable case: the $\ell_\infty$ norm, $\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$.

- Notable case: the $\ell_0$ “norm” (not): $\|x\|_0 = |\{i : x_i \neq 0\}|$. 
Special Matrices

- The identity matrix $I \in \mathbb{R}^{n \times n}$ is a square matrix such that

$$l_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

\[ I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \]
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- Diagonal matrix: $A \in \mathbb{R}^{n \times n}$ is diagonal if $(i \neq j) \Rightarrow A_{i,j} = 0$.
- Upper triangular matrix: $(j < i) \Rightarrow A_{i,j} = 0$. 
Special Matrices

• The identity matrix $I \in \mathbb{R}^{n \times n}$ is a square matrix such that

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• Neutral element of matrix product: $AI = IA = A$.

• Diagonal matrix: $A \in \mathbb{R}^{n \times n}$ is diagonal if $(i \neq j) \Rightarrow A_{i,j} = 0$.

• Upper triangular matrix: $(j < i) \Rightarrow A_{i,j} = 0$.

• Lower triangular matrix: $(j > i) \Rightarrow A_{i,j} = 0$. 
• A vector $x \in \mathbb{R}^n$ is an eigenvector of matrix $A \in \mathbb{R}^{n \times n}$ if

$$A x = \lambda x,$$

where $\lambda \in \mathbb{R}$ is the corresponding eigenvalue.
A vector $x \in \mathbb{R}^n$ is an eigenvector of matrix $A \in \mathbb{R}^{n \times n}$ if

$$Ax = \lambda x,$$

where $\lambda \in \mathbb{R}$ is the corresponding eigenvalue.

The eigenvalues of a diagonal matrix are the elements in the diagonal.
A vector \( x \in \mathbb{R}^n \) is an eigenvector of matrix \( A \in \mathbb{R}^{n \times n} \) if

\[
Ax = \lambda x,
\]

where \( \lambda \in \mathbb{R} \) is the corresponding eigenvalue.

The eigenvalues of a diagonal matrix are the elements in the diagonal.

Matrix trace:

\[
\text{trace}(A) = \sum_i A_{i,i} = \sum_i \lambda_i
\]
Eigenvalues, eigenvectors, determinant, trace

- A vector $x \in \mathbb{R}^n$ is an eigenvector of matrix $A \in \mathbb{R}^{n \times n}$ if
  \[ A x = \lambda x, \]
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- The eigenvalues of a diagonal matrix are the elements in the diagonal.
- Matrix trace:
  \[ \text{trace}(A) = \sum_i A_{i,i} = \sum_i \lambda_i \]
- Matrix determinant:
  \[ |A| = \det(A) = \prod_i \lambda_i \]
### Eigenvalues, eigenvectors, determinant, trace

- A vector \( x \in \mathbb{R}^n \) is an **eigenvector** of matrix \( A \in \mathbb{R}^{n \times n} \) if
  \[
  A x = \lambda x,
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  where \( \lambda \in \mathbb{R} \) is the corresponding **eigenvalue**.

- The eigenvalues of a diagonal matrix are the elements in the diagonal.

- Matrix **trace**:
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- Matrix **determinant**:
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- Properties: \( |AB| = |A||B| \),
Eigenvalues, eigenvectors, determinant, trace

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- Properties: $|AB| = |A||B|$, $|A^T| = |A|$. 
A vector $\mathbf{x} \in \mathbb{R}^n$ is an eigenvector of matrix $A \in \mathbb{R}^{n \times n}$ if

$$A \mathbf{x} = \lambda \mathbf{x},$$

where $\lambda \in \mathbb{R}$ is the corresponding eigenvalue.

The eigenvalues of a diagonal matrix are the elements in the diagonal.

Matrix trace:

$$\text{trace}(A) = \sum_i A_{i,i} = \sum_i \lambda_i$$

Matrix determinant:

$$|A| = \det(A) = \prod_i \lambda_i$$

Properties:

$$|AB| = |A||B|, \quad |A^T| = |A|, \quad |\alpha A| = \alpha^n |A|$$
Matrix Inverse

- Matrix $A \in \mathbb{R}^{n\times n}$ is invertible if there is $B \in \mathbb{R}^{n\times n}$ s.t. $AB = BA = I$. 
Matrix Inverse

- Matrix $A \in \mathbb{R}^{n \times n}$ is invertible if there is $B \in \mathbb{R}^{n \times n}$ s.t. $AB = BA = I$.

- ...matrix $B$, such that $AB = BA = I$, denoted $B = A^{-1}$. 
Matrix $A \in \mathbb{R}^{n \times n}$ is invertible if there is $B \in \mathbb{R}^{n \times n}$ s.t. $AB = BA = I$.

...matrix $B$, such that $AB = BA = I$, denoted $B = A^{-1}$.

Matrix $A \in \mathbb{R}^{n \times n}$ is invertible $\iff$ $\det(A) \neq 0$. 

Determinant of inverse: $\det(A^{-1}) = 1/\det(A)$. 

Solving system $Ax = b$, if $A$ is invertible: $x = A^{-1}b$. 

Properties: $(A^{-1})^{-1} = A$, $(A^{-1})^T = (A^T)^{-1}$, $(AB)^{-1} = B^{-1}A^{-1}$.

There are many algorithms to compute $A^{-1}$; general case, computational cost $O(n^3)$. 

André Martins (IST)
- Matrix $A \in \mathbb{R}^{n \times n}$ is invertible if there is $B \in \mathbb{R}^{n \times n}$ s.t. $AB = BA = I$.

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André Martins (IST)
Lecture 1
IST, Fall 2019 40 / 82
Matrix Inverse

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- Properties: $(A^{-1})^{-1} = A$.
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Quadratic Forms and Positive (Semi-)Definite Matrices

- Given matrix $A \in \mathbb{R}^{n \times n}$ and vector $x \in \mathbb{R}^n$, 
  
  $$x^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} x_i x_j \in \mathbb{R}$$

  is called a quadratic form.
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Convex and strictly convex sets

\[ S \text{ is convex if } x, x' \in S \implies \forall \lambda \in [0, 1], \; \lambda x + (1 - \lambda)x' \in S \]

\[ S \text{ is strictly convex if } x, x' \in S \implies \forall \lambda \in (0, 1), \; \lambda x + (1 - \lambda)x' \in \text{int}(S) \]
Convex and strictly convex functions

Extended real valued function: \( f : \mathbb{R}^N \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \)

Domain of a function: \( \text{dom}(f) = \{x : f(x) \neq +\infty\} \)

\( f \) is a convex function if
\[
\forall \lambda \in [0, 1], x, x' \in \text{dom}(f) \quad f(\lambda x + (1 - \lambda)x') \leq \lambda f(x) + (1 - \lambda)f(x')
\]

\( f \) is a strictly convex function if
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Outline

1 Introduction

2 Class Administrativia

3 Recap

   Linear Algebra

   Probability Theory

   Optimization
Probability theory

- "Essentially, all models are wrong, but some are useful"; G. Box, 1987
- The study of probability has roots in games of chance (dice, cards, ...)
- Great names in science: Cardano, Fermat, Pascal, Laplace, Gauss, Huygens, Legendre, Poisson, Kolmogorov, Bernoulli, Cauchy, Gibbs, Boltzman, de Finetti, ...
- Natural tool to model uncertainty, information, knowledge, belief, ...
- ...thus also learning, decision making, inference, ...

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What is probability?

• Classical definition: \( \mathbb{P}(A) = \frac{N_A}{N} \)

...with \( N \) mutually exclusive equally likely outcomes, \( N_A \) of which result in the occurrence of event \( A \). \( \text{Laplace, 1814} \)

Example: \( \mathbb{P}(\text{randomly drawn card is } \clubsuit) = \frac{13}{52}. \)

Example: \( \mathbb{P}(\text{getting 1 in throwing a fair die}) = \frac{1}{6}. \)
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...relative frequency of occurrence of \( A \) in infinite number of trials.

• Subjective probability: \( P(A) \) is a degree of belief.  

de Finetti, 1930s

...gives meaning to \( P(\text{“Tomorrow it will rain”}) \).
Key concepts: Sample space and events

- **Sample space** $\mathcal{X} =$ set of possible outcomes of a random experiment.

  **Examples:**
  - Tossing two coins: $\mathcal{X} = \{HH, TH, HT, TT\}$
  - Roulette: $\mathcal{X} = \{1, 2, \ldots, 36\}$
  - Draw a card from a shuffled deck: $\mathcal{X} = \{A\spadesuit, 2\spadesuit, \ldots, Q\diamond, K\diamond\}$. 
Key concepts: Sample space and events

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- **An event** \( A \) is a subset of \( X \): \( A \subseteq X \).
  
  **Examples:**
  - “exactly one H in 2-coin toss”: \( A = \{TH, HT\} \subset \{HH, TH, HT, TT\} \).
  - “odd number in the roulette”: \( B = \{1, 3, \ldots, 35\} \subset \{1, 2, \ldots, 36\} \).
  - “drawn a ♥ card”: \( C = \{A♥, 2♥, \ldots, K♥\} \subset \{A♣, \ldots, K♦\} \).
Kolmogorov’s Axioms for Probability

- Probability is a function that maps events $A$ into the interval $[0, 1]$.

Kolmogorov’s axioms (1933) for probability $\mathbb{P}$

- $P(A) \geq 0$
- $P(X) = 1$
- If $A_1, A_2, \ldots \subseteq X$ are disjoint events, then $P(\bigcup_i A_i) = \sum_i P(A_i)$

From these axioms, many results can be derived. Examples:

- $P(\emptyset) = 0$
- $C \subset D \Rightarrow P(C) \leq P(D)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B) \leq P(A) + P(B)$ (union bound)
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Conditional Probability and Independence

- If $P(B) > 0$, $P(A|B) = \frac{P(A \cap B)}{P(B)}$ (conditional prob. of $A$ given $B$)
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- ...satisfies all of Kolmogorov's axioms:
  - For any $A \subseteq X$, $P(A|B) \geq 0$
  - $P(X|B) = 1$
  - If $A_1, A_2, ... \subseteq X$ are disjoint, then $P\left(\bigcup A_i \bigg| B\right) = \sum_i P(A_i|B)$

- Events $A, B$ are independent ($A \perp \perp B$) $\iff P(A \cap B) = P(A) \cdot P(B)$. 
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- Events $A$, $B$ are independent ($\perp \perp$) $\iff$ $\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B)$. 
Conditional Probability and Independence

• If $\mathbb{P}(B) > 0$, $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$.

• Events $A, B$ are independent ($A \perp \perp B$) $\iff \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

• Relationship with conditional probabilities: $A \perp \perp B \iff \mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)}$.

• Example: $X = \text{"52 cards"}$, $A = \{3\heartsuit, 3\clubsuit, 3\diamondsuit, 3\spadesuit\}$, and $B = \{A\heartsuit, 2\heartsuit, \ldots, K\heartsuit\}$; then, $\mathbb{P}(A) = \frac{1}{13}$, $\mathbb{P}(B) = \frac{1}{4}$, $\mathbb{P}(A \cap B) = \mathbb{P}(\{3\heartsuit\}) = \frac{1}{52}$, $\mathbb{P}(A|B) = \mathbb{P}(\text{"3"|"♥"}) = \frac{1}{13} = \mathbb{P}(A)$.
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Example: $X = \{52$ cards$\}$, $A = \{3\text{♥}, 3\text{♣}, 3\text{♦}, 3\text{♠}\}$, and $B = \{A\text{♥}, 2\text{♥}, ..., K\text{♥}\}$; then, $\mathbb{P}(A) = \frac{1}{13}$, $\mathbb{P}(B) = \frac{1}{4}$, $\mathbb{P}(A \cap B) = \mathbb{P}(\{3\text{♥}\}) = \frac{1}{52}$, $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\frac{1}{52}}{\frac{1}{4}} = \frac{1}{13} = \mathbb{P}(A)$.
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• If \( P(B) > 0 \), \( P(A|B) = \frac{P(A \cap B)}{P(B)} \)

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- Example: \( \mathcal{X} = \text{“52 cards”} \), \( A = \{3\heartsuit, 3\spadesuit, 3\diamondsuit, 3\clubsuit\} \), and \( B = \{A\heartsuit, 2\heartsuit, \ldots, K\heartsuit\} \); then, \( P(A) = 1/13 \), \( P(B) = 1/4 \)

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\[
\begin{align*}
P(A \cap B) &= P(\{3\heartsuit\}) = \frac{1}{52} \\
P(A)P(B) &= \frac{1}{13} \frac{1}{4} = \frac{1}{52} \\
P(A|B) &= P(\text{“3”|“\diamondsuit”}) = \frac{1}{13} = P(A)
\end{align*}
\]
Bayes Theorem

- Law of total probability: if \(A_1, \ldots, A_n\) are a partition of \(\mathcal{X}\)

\[
P(B) = \sum_i P(B|A_i)P(A_i)
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\]
A (real) random variable (RV) is a function: \( X : \mathcal{X} \rightarrow \mathbb{R} \)

- Discrete RV: range of \( X \) is countable (e.g., \( \mathbb{N} \) or \( \{0, 1\} \))
- Continuous RV: range of \( X \) is uncountable (e.g., \( \mathbb{R} \) or \( [0, 1] \))

Example: number of heads in tossing two coins, \( X = \{HH, HT, TH, TT\} \), \( X(HH) = 2 \), \( X(HT) = X(TH) = 1 \), \( X(TT) = 0 \).

Range of \( X \) = \( \{0, 1, 2\} \).

Example: distance traveled by a tossed coin; range of \( X \) = \( \mathbb{R}^+ \).
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- **Continuous RV**: range of \( X \) is uncountable (e.g., \( \mathbb{R} \) or \( [0, 1] \))

### Example: distance traveled by a tossed coin; range of \( X = \mathbb{R}^+ \).
• A (real) random variable (RV) is a function: $X : X \rightarrow \mathbb{R}$

• **Discrete RV**: range of $X$ is countable (e.g., $\mathbb{N}$ or $\{0, 1\}$)

• **Continuous RV**: range of $X$ is uncountable (e.g., $\mathbb{R}$ or $[0, 1]$)

• **Example**: number of head in tossing two coins,
  $X = \{HH, HT, TH, TT\}$,
  $X(HH) = 2$, $X(HT) = X(TH) = 1$, $X(TT) = 0$.
  Range of $X = \{0, 1, 2\}$.

• **Example**: distance traveled by a tossed coin; range of $X = \mathbb{R}_+$. 
Random Variables: Distribution Function

- **Distribution function:** $F_X(x) = \mathbb{P}(\{\omega \in \mathcal{X} : X(\omega) \leq x\})$

Example: number of heads in tossing 2 coins; range($X$) = \{0, 1, 2\}.

- **Probability mass function** (discrete RV): $f_X(x) = \mathbb{P}(X = x)$, $F_X(x) = \sum_{x_i \leq x} f_X(x_i)$. 
Random Variables: Distribution Function

- **Distribution function**: \( F_X(x) = \mathbb{P}(\{\omega \in \mathcal{X} : X(\omega) \leq x\}) \)

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- Distribution function: \( F_X(x) = \mathbb{P}(\{\omega \in \mathcal{X} : X(\omega) \leq x\}) \)

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- Probability mass function (discrete RV): \( f_X(x) = \mathbb{P}(X = x) \),

\[
F_X(x) = \sum_{X(x_i)} f_X(x_i).
\]
Important Discrete Random Variables

- **Uniform**: $X \in \{x_1, \ldots, x_K\}$, pmf $f_X(x_i) = 1/K$. 

- **Bernoulli RV**: $X \in \{0, 1\}$, pmf $f_X(x) = \{p \iff x = 1, 1-p \iff x = 0\}$. Can be written compactly as $f_X(x) = p^x(1-p)^{1-x}$.

- **Binomial RV**: $X \in \{0, 1, \ldots, n\}$ (sum on $n$ Bernoulli RVs) $f_X(x) = \text{Binomial}(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$. Binomial coefficients ("$n$ choose $x$":) $\binom{n}{x} = \frac{n!}{x!(n-x)!}$. 
Important Discrete Random Variables

- **Uniform**: $X \in \{x_1, ..., x_K\}$, pmf $f_X(x_i) = 1/K$.

- **Bernoulli RV**: $X \in \{0, 1\}$, pmf $f_X(x) = \begin{cases} p & \iff x = 1 \\ 1 - p & \iff x = 0 \end{cases}$

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  Binomial coefficients ("$n$ choose $x$":)
  
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Important Discrete Random Variables

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  Binomial coefficients ("$n$ choose $x$"):

  $\binom{n}{x} = \frac{n!}{(n - x)! \cdot x!}$
More Important Discrete Random Variables

- **Geometric**\((p)\): \( X \in \mathbb{N} \), pmf \( f_X(x) = p(1 - p)^{x-1} \).
  (e.g., number of trials until the first success).
More Important Discrete Random Variables

- **Geometric**($p$): $X \in \mathbb{N}$, pmf $f_X(x) = p(1 - p)^{x-1}$.
  (e.g., number of trials until the first success).

- **Poisson**($\lambda$): $X \in \mathbb{N} \cup \{0\}$, pmf $f_X(x) = \frac{e^{-\lambda} \lambda^x}{x!}$

Notice that $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^\lambda$, thus $\sum_{x=0}^{\infty} f_X(x) = 1$.

“...probability of the number of independent occurrences in a fixed (time/space) interval if these occurrences have known average rate”

![Graph showing different Poisson distributions for different $\lambda$ values]
Random Variables: Distribution Function

- Distribution function: $F_X(x) = \mathbb{P}(\{\omega \in \mathcal{X} : X(\omega) \leq x\})$

\[
F_X(x) = \int_{-\infty}^{x} f_X(u) \, du,
\]
\[
P(X \in [c, d]) = \int_{c}^{d} f_X(x) \, dx.
\]
\[
P(X = x) = 0.
\]
Random Variables: Distribution Function

- **Distribution function:** \( F_X(x) = \mathbb{P}(\{\omega \in \mathcal{X} : X(\omega) \leq x\}) \)

- **Example:** continuous RV with uniform distribution on \([a, b]\).

\[
\begin{align*}
\mathcal{X} & \quad \{\omega : X(\omega) \leq x\} \\
X & \quad X(\omega) \leq x \\
\mathbb{R} & \\
\end{align*}
\]
Random Variables: Distribution Function

- Distribution function: \( F_X(x) = \mathbb{P}(\{\omega \in \mathcal{X} : X(\omega) \leq x\}) \)

- Example: continuous RV with uniform distribution on \([a, b]\).

- Probability density function (pdf, continuous RV): \( f_X(x) \)
Random Variables: Distribution Function

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- **Probability density function** (pdf, continuous RV): \( f_X(x) \)
  \[
  F_X(x) = \int_{-\infty}^{x} f_X(u) \, du, \quad \mathbb{P}(X \in [c, d]) = \int_{c}^{d} f_X(x) \, dx,
  \]
Random Variables: Distribution Function

- **Distribution function:** $F_X(x) = \mathbb{P}(\{\omega \in \mathcal{X} : X(\omega) \leq x\})$

  ![Diagram of distribution function]

- **Example:** continuous RV with uniform distribution on $[a, b]$.

  ![Graph of uniform distribution]

- **Probability density function (pdf, continuous RV):** $f_X(x)$

  $$F_X(x) = \int_{-\infty}^{x} f_X(u) \, du, \quad \mathbb{P}(X \in [c, d]) = \int_{c}^{d} f_X(x) \, dx, \quad \mathbb{P}(X = x) = 0$$
• **Uniform**: \( f_X(x) = \text{Uniform}(x; a, b) = \begin{cases} \frac{1}{b-a} & \iff x \in [a, b] \\ 0 & \iff x \notin [a, b] \end{cases} \)

(previous slide).
Important Continuous Random Variables

- **Uniform**: \( f_X(x) = \text{Uniform}(x; a, b) = \begin{cases} \frac{1}{b-a} & \iff x \in [a, b] \\ 0 & \iff x \notin [a, b] \end{cases} \)

(Previous slide).

- **Gaussian**: 
  \[ f_X(x) = \mathcal{N}(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x-\mu)^2}{2 \sigma^2}} \]
Important Continuous Random Variables

- **Uniform**: \( f_X(x) = \text{Uniform}(x; a, b) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{if } x \notin [a, b] \end{cases} \)

(Previous slide).

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- **Exponential**: \( f_X(x) = \text{Exp}(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \)
Expectation of Random Variables

- **Expectation**: 
  \[ E(X) = \begin{cases} 
  \sum_{i} x_i f_X(x_i) & X \in \{x_1, ..., x_K\} \subset \mathbb{R} \\
  \int_{-\infty}^{\infty} x f_X(x) \, dx & X \text{ continuous} 
  \end{cases} \]

- Example: Bernoulli, 
  \[ f_X(x) = p x (1-p) \] for \( x \in \{0, 1\} \).
  \[ E(X) = 0 \times (1-p) + 1 \times p = p. \]

- Example: Binomial,
  \[ f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \] for \( x \in \{0, ..., n\} \).
  \[ E(X) = np. \]

- Example: Gaussian,
  \[ f_X(x) = \mathcal{N}(x; \mu, \sigma^2). \]
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  \[
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  \]

- **Example:** Binomial, \( f_X(x) = \binom{n}{x} p^x (1 - p)^{n-x} \), for \( x \in \{0, \ldots, n\} \).
  \[
  \mathbb{E}(X) = n \cdot p.
  \]
Expectation of Random Variables

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• **Example**: Gaussian, \( f_X(x) = \mathcal{N}(x; \mu, \sigma^2) \).
  \[ \mathbb{E}(X) = \mu. \]

• **Linearity of expectation**: 
  \[ \mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y); \quad \mathbb{E}(\alpha X) = \alpha \mathbb{E}(X), \quad \alpha \in \mathbb{R} \]
Expectation of Functions of Random Variables

- \( E(g(X)) = \begin{cases} 
\sum_{i} g(x_i)f_X(x_i) & X \text{ discrete}, \ g(x_i) \in \mathbb{R} \\
\int_{-\infty}^{\infty} g(x)f_X(x) \, dx & X \text{ continuous}
\end{cases} \)
Expectation of Functions of Random Variables

\[ \mathbb{E}(g(X)) = \begin{cases} 
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- **Example:** variance, \( \text{var}(X) = \mathbb{E}\left( (X - \mathbb{E}(X))^2 \right) \)
Expectation of Functions of Random Variables

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- **Example**: variance, \( \text{var}(X) = E\left( (X - E(X))^2 \right) = E(X^2) - E(X)^2 \)
- **Example**: Bernoulli variance, \( E(X^2) = E(X) = p \)
Expectation of Functions of Random Variables

\[ \mathbb{E}(g(X)) = \begin{cases} 
\sum_{i} g(x_i) f_X(x_i) & \text{X discrete, } g(x_i) \in \mathbb{R} \\
\int_{-\infty}^{\infty} g(x) f_X(x) \, dx & \text{X continuous}
\end{cases} \]

- **Example:** variance, \( \text{var}(X) = \mathbb{E}\left( (X - \mathbb{E}(X))^2 \right) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \)

- **Example:** Bernoulli variance, \( \mathbb{E}(X^2) = \mathbb{E}(X) = p \), thus \( \text{var}(X) = p(1 - p) \). 

André Martins (IST) Lecture 1 IST, Fall 2019
Expectation of Functions of Random Variables

\[ \mathbb{E}(g(X)) = \begin{cases} 
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Expectation of Functions of Random Variables

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- **Example:** Gaussian variance, \( \mathbb{E}((X - \mu)^2) = \sigma^2 \).

- Probability as expectation of indicator, \( \mathbf{1}_A(x) = \begin{cases} 1 & \Leftrightarrow x \in A \\ 0 & \Leftrightarrow x \notin A \end{cases} \)

\[
\mathbb{P}(X \in A) = \int_A f_X(x)\,dx = \int \mathbf{1}_A(x)f_X(x)\,dx = \mathbb{E}(\mathbf{1}_A(X))
\]
Two (or More) Random Variables

• Joint pmf of two discrete RVs: \( f_{X,Y}(x, y) = \mathbb{P}(X = x \land Y = y) \).

  Extends trivially to more than two RVs.
Two (or More) Random Variables

- **Joint pmf** of two discrete RVs: \( f_{X,Y}(x,y) = \mathbb{P}(X = x \land Y = y) \).
  Extends trivially to more than two RVs.

- **Joint pdf** of two continuous RVs: \( f_{X,Y}(x,y) \), such that
  \[
  \mathbb{P}((X, Y) \in A) = \int \int_A f_{X,Y}(x,y) \, dx \, dy.
  \]
  Extends trivially to more than two RVs.

- **Independence**: \( X \perp \!
\!
\!
\!\perp Y \) \( \iff \) \( f_{X,Y}(x,y) = f_X(x)f_Y(y) \).
Two (or More) Random Variables

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\]
  Extends trivially to more than two RVs.

• Marginalization:  \( f_Y(y) = \begin{cases} 
\sum_x f_{X,Y}(x,y), & \text{if } X \text{ is discrete} \\
\int_{\infty}^{\infty} f_{X,Y}(x,y) \, dx, & \text{if } X \text{ continuous}
\end{cases} \)
Two (or More) Random Variables

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- **Marginalization:**
  \[
  f_Y(y) = \begin{cases}
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  \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx, & \text{if } X \text{ continuous}
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- **Joint pmf** of two discrete RVs: \( f_{X,Y}(x,y) = \mathbb{P}(X = x \land Y = y) \).

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\end{cases}
\]

- **Independence:**

\( X \independent Y \iff f_{X,Y}(x,y) = f_X(x) f_Y(y) \implies \mathbb{E}(X Y) = \mathbb{E}(X) \mathbb{E}(Y). \)
Conditionals and Bayes’ Theorem

• **Conditional pmf** (discrete RVs):
  \[
  f_{X|Y}(x|y) = \mathbb{P}(X = x|Y = y) = \frac{\mathbb{P}(X = x \land Y = y)}{\mathbb{P}(Y = y)} = \frac{f_{X,Y}(x,y)}{f_Y(y)}.
  \]
**Conditionals and Bayes’ Theorem**

- **Conditional pmf (discrete RVs):**
  \[ f_{X \mid Y}(x \mid y) = \Pr(X = x \mid Y = y) = \frac{\Pr(X = x \land Y = y)}{\Pr(Y = y)} = \frac{f_{X,Y}(x,y)}{f_Y(y)}. \]

- **Conditional pdf (continuous RVs):**
  \[ f_{X \mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \]
  ...the meaning is technically delicate.
• **Conditional pmf** (discrete RVs):

\[ f_{X|Y}(x|y) = \mathbb{P}(X = x | Y = y) = \frac{\mathbb{P}(X = x \land Y = y)}{\mathbb{P}(Y = y)} = \frac{f_{X,Y}(x,y)}{f_Y(y)}. \]

• **Conditional pdf** (continuous RVs): \( f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \)

...the meaning is technically delicate.

• **Bayes’ theorem**: \( f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)} \) (pdf or pmf).
Conditionals and Bayes’ Theorem

- **Conditional pmf** (discrete RVs):
  \[ f_{X|Y}(x|y) = P(X = x|Y = y) = \frac{P(X = x \land Y = y)}{P(Y = y)} = \frac{f_{X,Y}(x,y)}{f_Y(y)}. \]

- **Conditional pdf** (continuous RVs):
  \[ f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \]
  ...the meaning is technically delicate.

- **Bayes’ theorem**:
  \[ f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)} \quad \text{(pdf or pmf)}. \]

- Also valid in the mixed case (e.g., \(X\) continuous, \(Y\) discrete).
Joint, Marginal, and Conditional Probabilities: An Example

- A pair of binary variables $X, Y \in \{0, 1\}$, with joint pmf:

<table>
<thead>
<tr>
<th>$f_{X,Y}(x, y)$</th>
<th>$Y = 0$</th>
<th>$Y = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X = 0$</td>
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Marginals:

- $f_X(0) = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$,
- $f_X(1) = \frac{1}{10} + \frac{3}{10} = \frac{4}{10}$,
- $f_Y(0) = \frac{1}{5} + \frac{1}{10} = \frac{3}{10}$,
- $f_Y(1) = \frac{2}{5} + \frac{3}{10} = \frac{7}{10}$. 
Joint, Marginal, and Conditional Probabilities: An Example

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</tr>
<tr>
<td>$X = 1$</td>
<td>1/10</td>
<td>3/10</td>
</tr>
</tbody>
</table>

• Marginals: $f_X(0) = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$, $f_X(1) = \frac{1}{10} + \frac{3}{10} = \frac{4}{10}$,

$f_Y(0) = \frac{1}{5} + \frac{1}{10} = \frac{3}{10}$, $f_Y(1) = \frac{2}{5} + \frac{3}{10} = \frac{7}{10}$.

• Conditional probabilities:

| $f_{X|Y}(x|y)$ | $Y = 0$ | $Y = 1$ |
|----------------|--------|--------|
| $X = 0$        | 2/3    | 4/7    |
| $X = 1$        | 1/3    | 3/7    |

| $f_{Y|X}(y|x)$ | $Y = 0$ | $Y = 1$ |
|----------------|--------|--------|
| $X = 0$        | 1/3    | 2/3    |
| $X = 1$        | 1/4    | 3/4    |
• Multinomial: \( X = (X_1, \ldots, X_K), X_i \in \{0, \ldots, n\}, \) such that \( \sum_i X_i = n, \)

\[
f_X(x_1, \ldots, x_K) = \begin{cases} \binom{n}{x_1 \ x_2 \ \cdots \ x_K} \prod_{i=1}^{K} p_i^{x_i} & \iff \sum_i x_i = n \\ 0 & \iff \sum_i x_i \neq n \end{cases}
\]

\[
\binom{n}{x_1 \ x_2 \ \cdots \ x_K} = \frac{n!}{x_1! \ x_2! \ \cdots \ x_K!}
\]

Parameters: \( p_1, \ldots, p_K \geq 0, \) such that \( \sum_i p_i = 1. \)
An Important Multivariate RV: Multinomial

- **Multinomial**: Let $X = (X_1, ..., X_K)$, $X_i \in \{0, ..., n\}$, such that $\sum_i X_i = n$.

\[
f_X(x_1, ..., x_K) = \begin{cases} \binom{n}{x_1 x_2 \cdots x_K} p_1^{x_1} p_2^{x_2} \cdots p_K^{x_K} & \iff \sum_i x_i = n \\ 0 & \iff \sum_i x_i \neq n \end{cases}
\]

\[
\binom{n}{x_1 x_2 \cdots x_K} = \frac{n!}{x_1! x_2! \cdots x_K!}
\]

Parameters: $p_1, ..., p_K \geq 0$, such that $\sum_i p_i = 1$.

- Generalizes the binomial from binary to $K$-classes.
An Important Multivariate RV: Multinomial

- **Multinomial:** $X = (X_1, \ldots, X_K)$, $X_i \in \{0, \ldots, n\}$, such that $\sum_i X_i = n$,

$$f_X(x_1, \ldots, x_K) = \begin{cases} \binom{n}{x_1 \ x_2 \ \cdots \ x_K} p_1^{x_1} p_2^{x_2} \cdots p_K^{x_K} \quad \Leftarrow \sum_i x_i = n \\ 0 \quad \Leftarrow \sum_i x_i \neq n \end{cases}$$

\[
\binom{n}{x_1 \ x_2 \ \cdots \ x_K} = \frac{n!}{x_1! \ x_2! \ \cdots \ x_K!}
\]

Parameters: $p_1, \ldots, p_K \geq 0$, such that $\sum_i p_i = 1$.

- Generalizes the binomial from binary to $K$-classes.

- **Example:** tossing $n$ independent fair dice, $p_1 = \cdots = p_6 = 1/6$.

  $x_i =$ number of outcomes with $i$ dots. Of course, $\sum_i x_i = n$. 
• Multivariate Gaussian: \( X \in \mathbb{R}^n \),

\[
f_X(x) = \mathcal{N}(x; \mu, C) = \frac{1}{\sqrt{\det(2\pi C)}} \exp \left( -\frac{1}{2} (x - \mu)^T C^{-1} (x - \mu) \right)
\]
An Important Multivariate RV: Gaussian

- Multivariate Gaussian: $X \in \mathbb{R}^n$,

$$f_X(x) = \mathcal{N}(x; \mu, C) = \frac{1}{\sqrt{\det(2\pi C)}} \exp \left( -\frac{1}{2}(x - \mu)^T C^{-1}(x - \mu) \right)$$

- Parameters: vector $\mu \in \mathbb{R}^n$ and matrix $C \in \mathbb{R}^{n \times n}$.
  Expected value: $\mathbb{E}(X) = \mu$. Meaning of $C$: next slide.
An Important Multivariate RV: Gaussian

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- **Parameters:** vector $\mu \in \mathbb{R}^n$ and matrix $C \in \mathbb{R}^{n \times n}$. Expected value: $\mathbb{E}(X) = \mu$. Meaning of $C$: next slide.
• Covariance between two RVs:

\[
\text{cov}(X, Y) = \mathbb{E} \left[ (X - \mathbb{E}(X)) (Y - \mathbb{E}(Y)) \right] = \mathbb{E}(X Y) - \mathbb{E}(X) \mathbb{E}(Y)
\]
Covariance, Correlation, and all that...

- **Covariance** between two RVs:

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\]

- Relationship with variance: \( \text{var}(X) = \text{cov}(X, X) \).
• **Covariance** between two RVs:

\[
\text{cov}(X, Y) = \mathbb{E}
\left[
(X - \mathbb{E}(X)) \ (Y - \mathbb{E}(Y))
\right]
= \mathbb{E}(X \ Y) - \mathbb{E}(X) \mathbb{E}(Y)
\]

• **Relationship with variance:** \( \text{var}(X) = \text{cov}(X, X) \).

• **Correlation:** 

\[
\text{corr}(X, Y) = \rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X) \cdot \text{var}(Y)}} \in [-1, 1]
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• \( X \perp \perp Y \iff f_{X,Y}(x, y) = f_X(x) f_Y(y) \)
**Covariance, Correlation, and all that...**

- **Covariance** between two RVs:
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  \text{cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\right] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)
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- **Correlation**: \(\text{corr}(X, Y) = \rho(X, Y) = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\text{var}(Y)}} \in [-1, 1]\)

- \(X \perp \!\!\!\! \perp Y \iff f_{X,Y}(x,y) = f_X(x)f_Y(y) \Rightarrow \text{cov}(X, Y) = 0\) (example)

- **Covariance matrix** of multivariate RV, \(X \in \mathbb{R}^n\):
  \[
  \text{cov}(X) = \mathbb{E}\left[(X - \mathbb{E}(X))(X - \mathbb{E}(X))^T\right] = \mathbb{E}(XX^T) - \mathbb{E}(X)\mathbb{E}(X)^T
  \]

- **Covariance of Gaussian RV**, \(f_X(x) = \mathcal{N}(x; \mu, \Sigma)\): \(\Rightarrow \text{cov}(X) = \Sigma\)
Covariance, Correlation, and all that...

- **Covariance** between two RVs:
  \[ \text{cov}(X, Y) = \mathbb{E} \left[ (X - \mathbb{E}(X))(Y - \mathbb{E}(Y)) \right] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \]

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Covariance, Correlation, and all that...

**Covariance** between two RVs:

\[ \text{cov}(X, Y) = \mathbb{E} \left[ (X - \mathbb{E}(X)) (Y - \mathbb{E}(Y)) \right] = \mathbb{E}(X Y) - \mathbb{E}(X) \mathbb{E}(Y) \]

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\( X \perp \perp Y \iff f_{X,Y}(x, y) = f_X(x) f_Y(y) \quad \implies \quad \text{cov}(X, Y) = 0 \) (example)

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\[ \text{cov}(X) = \mathbb{E} \left[ (X - \mathbb{E}(X))(X - \mathbb{E}(X))^T \right] = \mathbb{E}(XX^T) - \mathbb{E}(X)\mathbb{E}(X)^T \]

**Covariance of Gaussian RV,** \( f_X(x) = \mathcal{N}(x; \mu, C) \) \( \Rightarrow \) \( \text{cov}(X) = C \)
Let $A \in \mathbb{R}^{n \times n}$ be a matrix and $a \in \mathbb{R}^n$ a vector.
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- If $\mathbb{E}(X) = \mu$ and $Y = AX$, then $\mathbb{E}(Y) = A\mu$;
Let $A \in \mathbb{R}^{n \times n}$ be a matrix and $a \in \mathbb{R}^n$ a vector.

- If $\mathbb{E}(X) = \mu$ and $Y = AX$, then $\mathbb{E}(Y) = A\mu$;
- If $\text{cov}(X) = C$ and $Y = AX$, then $\text{cov}(Y) = ACA^T$;
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- If \( \text{cov}(X) = C \) and \( Y = a^T X \in \mathbb{R} \), then \( \text{var}(Y) = a^T Ca \geq 0 \);
More on Expectations and Covariances

Let $A \in \mathbb{R}^{n \times n}$ be a matrix and $a \in \mathbb{R}^n$ a vector.

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More on Expectations and Covariances

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- If \( f_X(x) = \mathcal{N}(x; \mu, C) \) and \( Y = C^{-1/2}(X - \mu) \), then \( f_Y(y) = \mathcal{N}(y; 0, I) \).
Central Limit Theorem

Take $n$ independent r.v. $X_1, ..., X_n$ such that $\mathbb{E}[X_i] = \mu_i$ and $\text{var}(X_i) = \sigma_i^2$. 

Thus, if $Z_n = Y_n - \mu_i / \sigma_i$,

$$
\mathbb{E}[Z_n] = 0 \quad \text{and} \quad \text{var}(Z_n) = 1
$$

Central limit theorem (CLT): under some mild conditions on $X_1, ..., X_n$,

$$
\lim_{n \to \infty} Z_n \sim N(0, 1)
$$
Central Limit Theorem

Take \( n \) independent r.v. \( X_1, \ldots, X_n \) such that \( \mathbb{E}[X_i] = \mu_i \) and \( \text{var}(X_i) = \sigma_i^2 \)

- Their sum, \( Y_n = \sum_{i=1}^{n} X_i \) satisfies:

\[
\mathbb{E}[Y_n] = \sum_{i=1}^{n} \mu_i \equiv \mu
\]
Central Limit Theorem

Take $n$ independent r.v. $X_1, \ldots, X_n$ such that $\mathbb{E}[X_i] = \mu_i$ and $\text{var}(X_i) = \sigma_i^2$.

- Their sum, $Y_n = \sum_{i=1}^{n} X_i$, satisfies:
  
  $$\mathbb{E}[Y_n] = \sum_{i=1}^{n} \mu_i \equiv \mu$$
  $$\text{var}(Y_n) = \sum_{i} \sigma_i^2 \equiv \sigma$$
Central Limit Theorem

Take \( n \) independent r.v. \( X_1, \ldots, X_n \) such that \( \mathbb{E}[X_i] = \mu_i \) and \( \text{var}(X_i) = \sigma_i^2 \)

- Their sum, \( Y_n = \sum_{i=1}^{n} X_i \) satisfies:

\[
\mathbb{E}[Y_n] = \sum_{i=1}^{n} \mu_i \equiv \mu \quad \text{var}(Y_n) = \sum_{i} \sigma_i^2 \equiv \sigma
\]

- ...thus, if \( Z_n = \frac{Y_n - \mu}{\sigma} \)

\[
\mathbb{E}[Z_n] = 0 \quad \text{var}(Z_n) = 1
\]

- Central limit theorem (CLT): under some mild conditions on \( X_1, \ldots, X_n \)

\[
\lim_{n \to \infty} Z_n \sim \mathcal{N}(0, 1)
\]
Illustration

Sum of two i.i.d variables from a uniform in [-1,1]

Sum of twenty five i.i.d r.v from a uniform in [-1,1]
Important Inequalities

- **Markov’s inequality:** if $X \geq 0$ is an RV with expectation $\mathbb{E}(X)$, then

$$\mathbb{P}(X > t) \leq \frac{\mathbb{E}(X)}{t}$$

Simple proof:

$$t \mathbb{P}(X > t) = \int_{t}^{\infty} t f_X(x) \, dx \leq \int_{t}^{\infty} x f_X(x) \, dx = \mathbb{E}(X) - \int_{0}^{t} x f_X(x) \, dx = \mathbb{E}(X) - 0$$

- **Chebyshev’s inequality:** if $\mu = \mathbb{E}(Y)$ and $\sigma^2 = \text{var}(Y)$, then

$$\mathbb{P}(|Y - \mu| \geq s) \leq \frac{\sigma^2}{s^2}$$

...simple corollary of Markov’s inequality, with $X = |Y - \mu|^2$, $t = s^2$.
Important Inequalities

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Important Inequalities

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- **Chebyshev’s inequality:** $\mu = \mathbb{E}(Y)$ and $\sigma^2 = \text{var}(Y)$, then

$$\mathbb{P}(|Y - \mu| \geq s) \leq \frac{\sigma^2}{s^2}$$

...simple corollary of Markov’s inequality, with $X = |Y - \mu|^2$, $t = s^2$
Other Important Inequalities: Cauchy-Schwartz

- Cauchy-Schwartz’s inequality for RVs:
  \[ |E(XY)| \leq \sqrt{E(X^2)E(Y^2)} \]
Other Important Inequalities: Cauchy-Schwartz

- Cauchy-Schwartz’s inequality for RVs:

\[ |\mathbb{E}(X Y)| \leq \sqrt{\mathbb{E}(X^2) \mathbb{E}(Y^2)} \]

...why? Because \( \langle X, Y \rangle \equiv \mathbb{E}[XY] \) is a valid inner product
Cauchy-Schwartz’s inequality for RVs:

$$|\mathbb{E}(XY)| \leq \sqrt{\mathbb{E}(X^2) \mathbb{E}(Y^2)}$$

...why? Because $\langle X, Y \rangle \equiv \mathbb{E}[XY]$ is a valid inner product

Important corollary: let $\mathbb{E}[X] = \mu$ and $\mathbb{E}[X] = \nu$, 

$$|\text{cov}(X,Y)| = \mathbb{E}[(X-\mu)(Y-\nu)] \leq \sqrt{\mathbb{E}[(X-\mu)^2]} \sqrt{\mathbb{E}[(Y-\nu)^2]} = \sqrt{\text{var}(X) \text{var}(Y)}$$

Implication for correlation:

$$\text{corr}(X,Y) = \rho(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X) \text{var}(Y)}} \in [-1,1]$$
Other Important Inequalities: Cauchy-Schwartz

- **Cauchy-Schwartz’s inequality** for RVs:
  \[ |\mathbb{E}(X Y)| \leq \sqrt{\mathbb{E}(X^2) \mathbb{E}(Y^2)} \]

  ...why? Because \( \langle X, Y \rangle \equiv \mathbb{E}[XY] \) is a valid inner product

- Important corollary: let \( \mathbb{E}[X] = \mu \) and \( \mathbb{E}[X] = \nu \),
  \[ |\text{cov}(X, Y)| = \mathbb{E}[(X - \mu)(Y - \nu)] \leq \sqrt{\mathbb{E}[(X - \mu)^2] \mathbb{E}[(Y - \nu)^2]} \]
  \[ = \sqrt{\text{var}(X) \text{var}(Y)} \]
Other Important Inequalities: Cauchy-Schwartz

- Cauchy-Schwartz’s inequality for RVs:
  \[ |E(X Y)| \leq \sqrt{E(X^2)E(Y^2)} \]
  ...why? Because \( \langle X, Y \rangle \equiv E[XY] \) is a valid inner product

- Important corollary: let \( E[X] = \mu \) and \( E[X] = \nu \),
  \[ |\text{cov}(X, Y)| = E[(X - \mu)(Y - \nu)] \]
  \[ \leq \sqrt{E[(X - \mu)^2]E[(Y - \nu)^2]} \]
  \[ = \sqrt{\text{var}(X) \text{var}(Y)} \]

- Implication for correlation:
  \[ \text{corr}(X, Y) = \rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}} \in [-1, 1] \]
• Recall that a real function $g$ is convex if, for any $x, y$, and $\alpha \in [0, 1]$

$$g(\alpha x + (1 - \alpha)y) \leq \alpha g(x) + (1 - \alpha)g(y)$$
Recall that a real function \( g \) is convex if, for any \( x, y, \) and \( \alpha \in [0, 1] \)
\[
g(\alpha x + (1 - \alpha)y) \leq \alpha g(x) + (1 - \alpha)g(y)
\]

**Jensen’s inequality:** if \( g \) is a real convex function, then
\[
g(\mathbb{E}(X)) \leq \mathbb{E}(g(X))
\]
• Recall that a real function $g$ is convex if, for any $x, y$, and $\alpha \in [0, 1]$

$$g(\alpha x + (1 - \alpha) y) \leq \alpha g(x) + (1 - \alpha) g(y)$$

Jensen’s inequality: if $g$ is a real convex function, then

$$g(\mathbb{E}(X)) \leq \mathbb{E}(g(X))$$

Examples: $\mathbb{E}(X)^2 \leq \mathbb{E}(X^2) \Rightarrow \text{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \geq 0$.

$$\mathbb{E}(\log X) \leq \log \mathbb{E}(X), \quad \text{for } X \text{ a positive RV.}$$
Entropy of a discrete RV $X \in \{1, \ldots, K\}$:
\[
H(X) = - \sum_{x=1}^{K} f_X(x) \log f_X(x)
\]
Entropy and all that...

Entropy of a discrete RV $X \in \{1, ..., K\}$:

$$H(X) = - \sum_{x=1}^{K} f_X(x) \log f_X(x)$$

- **Positivity:** $H(X) \geq 0$;
  $H(X) = 0 \iff f_X(i) = 1$, for exactly one $i \in \{1, ..., K\}$. 

- **Upper bound:** $H(X) \leq \log K$;
  $H(X) = \log K \iff f_X(x) = 1/K$, for all $x \in \{1, ..., K\}$. 

- **Measure of uncertainty/randomness of $X$.**

  - **Continuous RV $X$:** differential entropy:
    $$h(X) = -\int f_X(x) \log f_X(x) \, dx$$
    $h(X)$ can be positive or negative. Example, if $f_X(x) = \text{Uniform}(x; a, b)$, $h(X) = \log(b-a)$.

    - If $f_X(x) = \mathcal{N}(x; \mu, \sigma^2)$, then $h(X) = \frac{1}{2} \log(2\pi e \sigma^2)$.

    - If $\text{var}(Y) = \sigma^2$, then $h(Y) \leq \frac{1}{2} \log(2\pi e \sigma^2)$. 

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Entropy of a discrete RV $X \in \{1, ..., K\}$:

$$H(X) = - \sum_{x=1}^{K} f_X(x) \log f_X(x)$$

- **Positivity:** $H(X) \geq 0$;
  $$H(X) = 0 \iff f_X(i) = 1, \text{ for exactly one } i \in \{1, ..., K\}.$$

- **Upper bound:** $H(X) \leq \log K$;
  $$H(X) = \log K \iff f_X(x) = 1/k, \text{ for all } x \in \{1, ..., K\}$$
Entropy of a discrete RV $X \in \{1, \ldots, K\}$:

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Entropy of a discrete RV $X \in \{1, \ldots, K\}$: 

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  $H(X) = \log K \iff f_X(x) = 1/k$, for all $x \in \{1, \ldots, K\}$

- **Measure of uncertainty/randomness** of $X$

Continuous RV $X$, differential entropy: 

$$h(X) = - \int f_X(x) \log f_X(x) \, dx$$
Entropy of a discrete RV $X \in \{1, \ldots, K\}$: 
\[ H(X) = - \sum_{x=1}^{K} f_X(x) \log f_X(x) \]

- **Positivity:** $H(X) \geq 0$;
  
  \[ H(X) = 0 \iff f_X(i) = 1, \text{ for exactly one } i \in \{1, \ldots, K\}. \]

- **Upper bound:** $H(X) \leq \log K$;
  
  \[ H(X) = \log K \iff f_X(x) = 1/k, \text{ for all } x \in \{1, \ldots, K\} \]

- **Measure of uncertainty/randomness** of $X$

Continuous RV $X$, **differential entropy**:
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- $h(X)$ can be positive or negative. Example, if $f_X(x) = \text{Uniform}(x; a, b)$, $h(X) = \log(b - a)$. 

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Entropy and all that...

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Kullback-Leibler divergence (KLD) between two pmf:

\[ D(f_X \parallel g_X) = \sum_{x=1}^{K} f_X(x) \log \frac{f_X(x)}{g_X(x)} \]

Positivity:

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D(f_X \parallel g_X) = 0 \iff f_X(x) = g_X(x), \text{ for } x \in \{1, ..., K\}
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Kullback-Leibler divergence (KLD) between two pmf:

$$D(f_X \| g_X) = \sum_{x=1}^{K} f_X(x) \log \frac{f_X(x)}{g_X(x)}$$

Positivity: $D(f_X \| g_X) \geq 0$

$D(f_X \| g_X) = 0 \Leftrightarrow f_X(x) = g_X(x), \text{ for } x \in \{1, \ldots, K\}$

KLD between two pdf:

$$D(f_X \| g_X) = \int f_X(x) \log \frac{f_X(x)}{g_X(x)} \, dx$$
Kullback-Leibler divergence (KLD) between two pmf:

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KLD between two pdf:

$$D(f_X \parallel g_X) = \int f_X(x) \log \frac{f_X(x)}{g_X(x)} \, dx$$

Positivity: $D(f_X \parallel g_X) \geq 0$

$D(f_X \parallel g_X) = 0 \iff f_X(x) = g_X(x)$, almost everywhere
Mutual information (MI) between two random variables:

\[ I(X; Y) = D(f_{X,Y} \| f_X f_Y) \]
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Positivity: $$I(X; Y) \geq 0$$

$$I(X; Y) = 0 \iff X, Y \text{ are independent.}$$

MI is a measure of dependency between two random variables.
Recommended Reading

Outline

1 Introduction

2 Class Administrativia

3 Recap

   Linear Algebra

   Probability Theory

   Optimization
Minimizing a function

• We are given a function $f : \mathbb{R}^n \to \mathbb{R}$.

• Goal: find $x^*$ that minimizes $f : \mathbb{R}^n \to \mathbb{R}$.

• Global minimum: for any $x \in \mathbb{R}^n$, $f(x^*) \leq f(x)$.

• Local minimum: for any $\|x - x^*\| \leq \delta \Rightarrow f(x^*) \leq f(x)$.

Are these global minima?

• No, (local minima, saddle points, ... )
Iterative descent methods

Goal: find the minimum/minimizer of $f : \mathbb{R}^d \to \mathbb{R}$

- Proceed in **small steps** in the **optimal direction** till a **stopping criterion** is met.
- **Gradient descent**: updates of the form: $x^{(t+1)} \leftarrow x^{(t)} - \eta(t) \nabla f(x^{(t)})$

**Figure:** Illustration of gradient descent. The blue circles correspond to the function values at different points, while the red lines correspond to steps taken in the negative gradient direction.
Convex functions

Pro: Guarantee of a global minima ✓

Figure: Illustration of a convex function. The line segment between any two points on the graph lies entirely above the curve.
Non-Convex functions

Pro: No guarantee of a global minima $\mathbf{x}$

Figure: Illustration of a non-convex function. Note the line segment intersecting the curve.
Thank you!

Questions?