Lecture 1: Introduction

André Martins

Deep Structured Learning Course, Fall 2020

Course Website

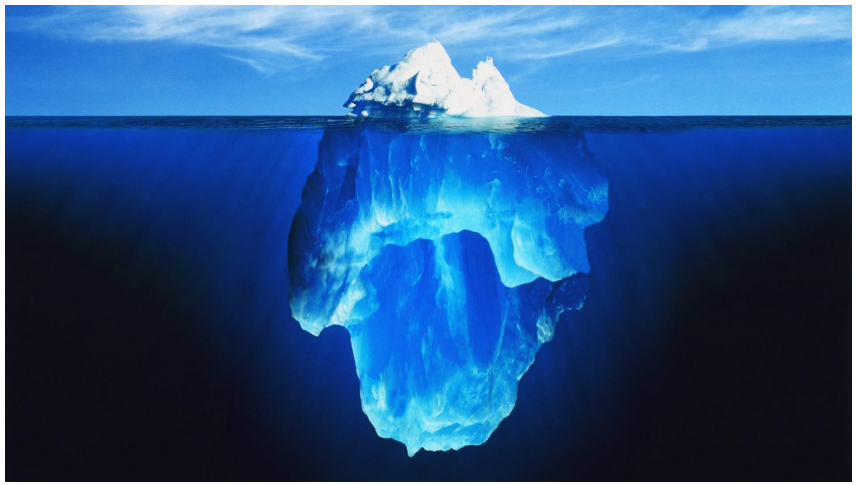
`https://andre-martins.github.io/pages/deep-structured-learning-ist-fall-2020.html`

There I'll post:

- Syllabus
- Lecture slides
- Literature pointers
- Homework assignments
- ...

Outline

What is "“Deep Learning”?



- Neural networks?
- Neural networks with many hidden layers?
- Anything beyond shallow (linear) models for statistical learning?
- Anything that learns representations?
- A form of learning that is really intense and profound?

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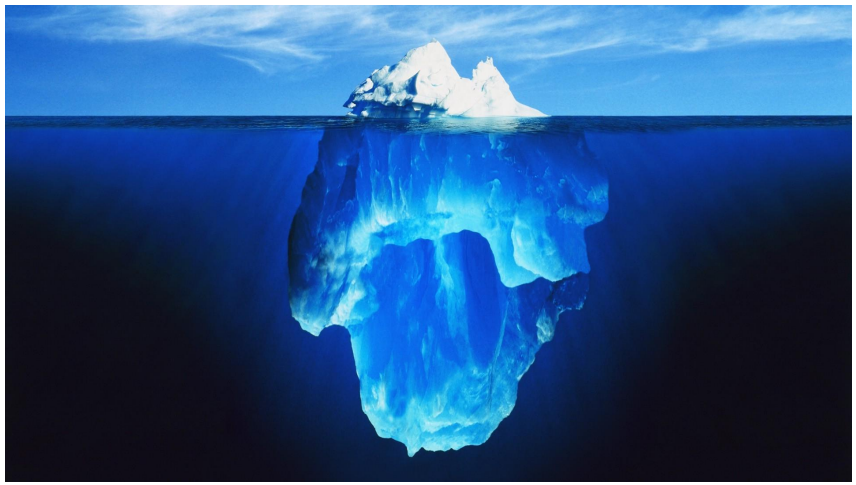

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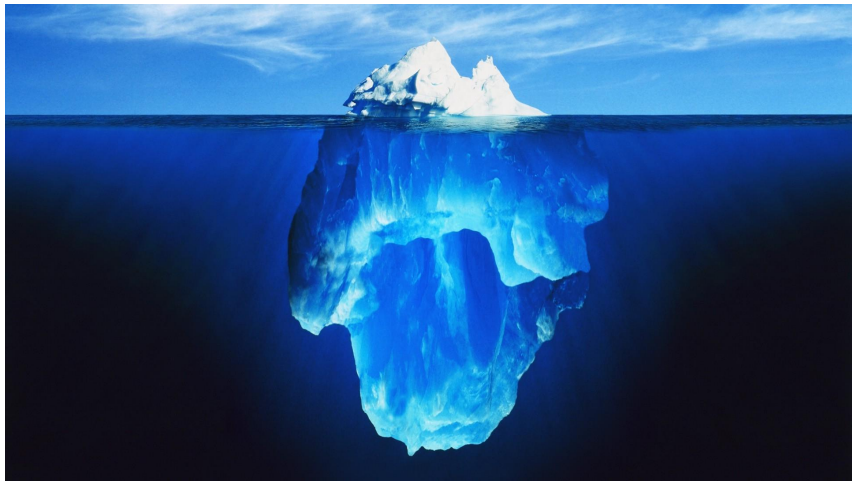
Where is the "Structure"?

- In the **input** objects (text, graphs, images, ...)
- In the **outputs** we want to predict (parsing, graph labeling, image segmentation, ...)
- In our **model** (convolutional networks, attention mechanisms)
- Related: **latent structure** (typically a way of encoding prior knowledge into the model)

This Course: "Deep Learning + Structure"



Why Did Deep Learning Become Mainstream?

Lots of recent breakthroughs:

- Object recognition
- Speech and language processing
- Chatbots and dialog systems
- Self-driving cars
- Machine translation
- Solving games (Atari, Go)

No signs of slowing down...


smart
On my way

Microsoft's Deep Learning Project Outperforms Humans In Image Recognition

Michael Thomsen, CONTRIBUTOR

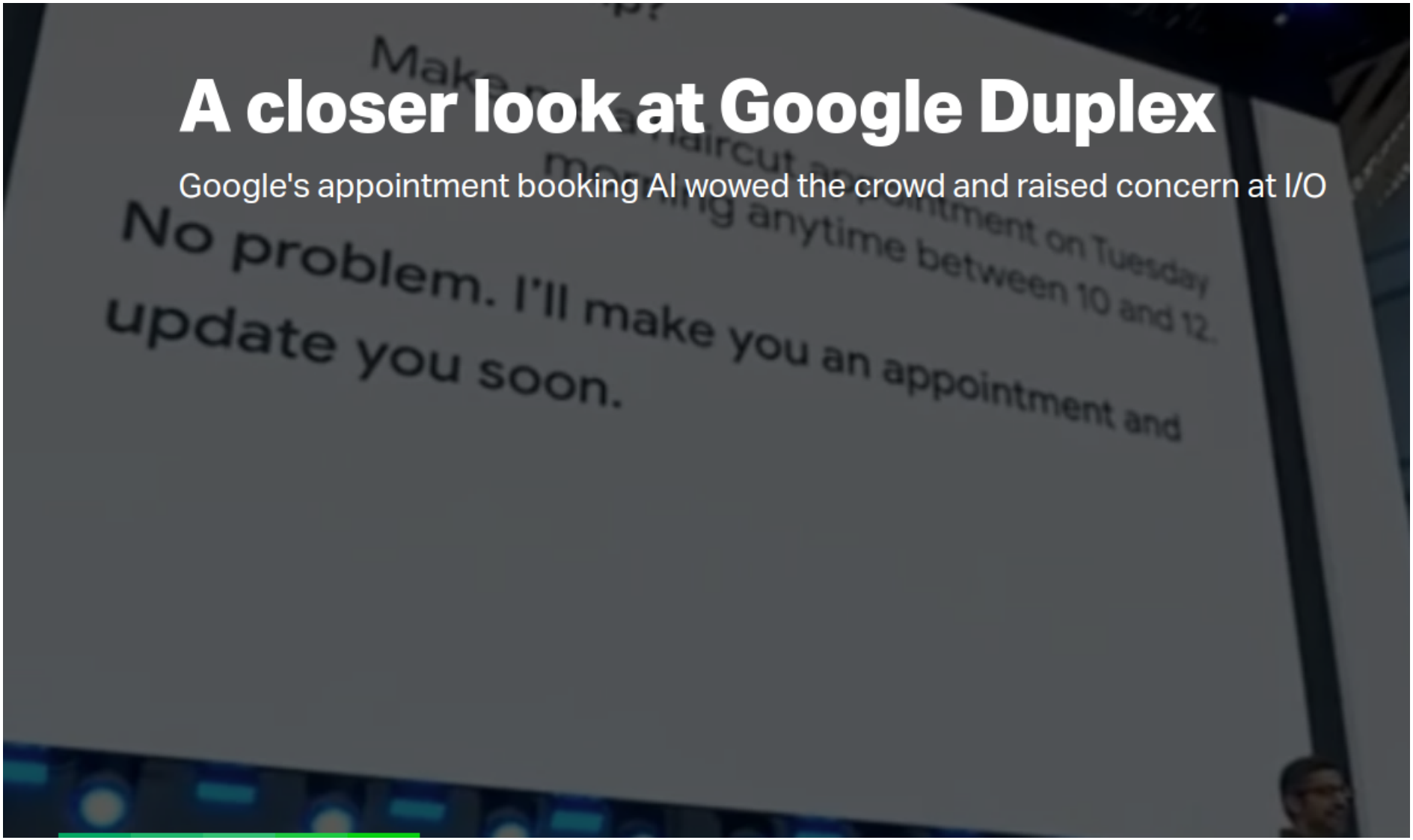
I write about tech, video games, science and culture. **FULL BIO**

Opinions expressed by Forbes Contributors are their own.

Microsoft's new breakthrough: AI that's as good as humans at listening... on the phone

Microsoft's new speech-recognition record means professional transcribers could be among the first to lose their jobs to artificial intelligence.

By Liam Tung | October 19, 2016 -- 10:10 GMT (11:10 BST) | Topic: Innovation


A closer look at Google Duplex
Google's appointment booking AI wowed the crowd and raised concern at I/O
No problem. I'll make you an appointment and update you soon.

Who is wearing glasses?

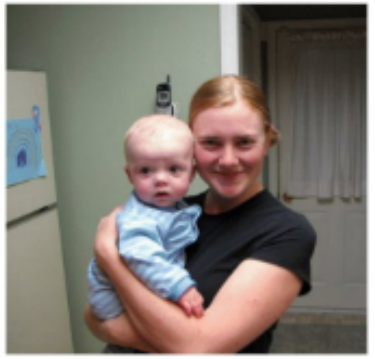
man

woman

Where is the child sitting?

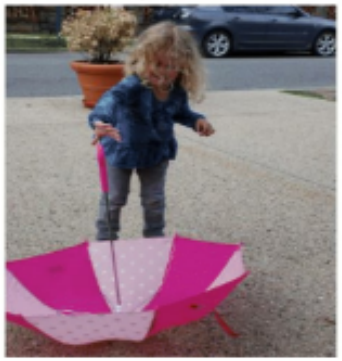
fridge

arms



Is the umbrella upside down?

yes



no

How many children are in the bed?

2

1

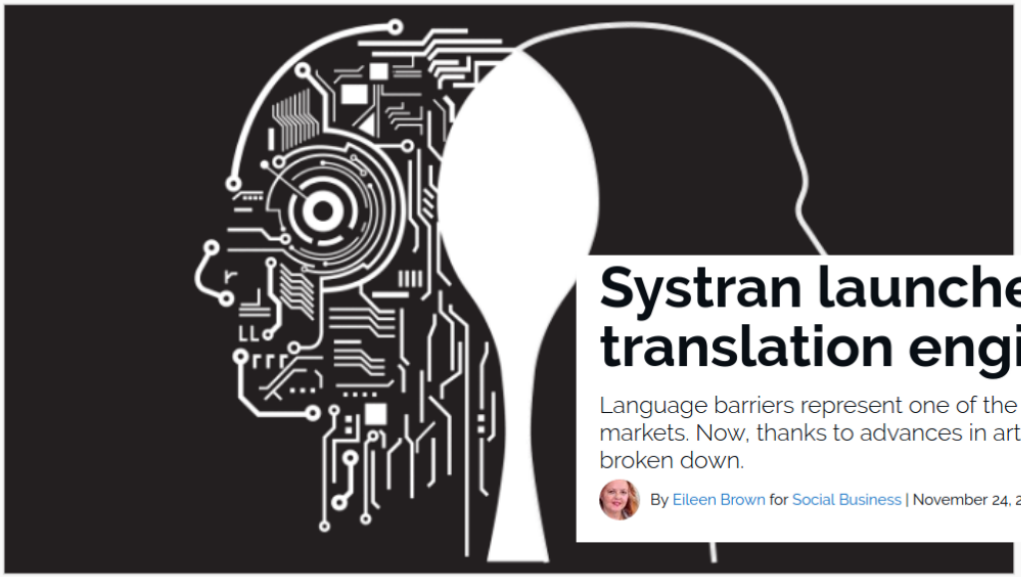
The Great A.I. Awakening

How Google used artificial intelligence to transform Google Translate, one of its more popular services — and how machine learning is poised to reinvent computing itself.

BY GIDEON LEWIS-KRAUS DEC. 14, 2016

Google unleashes deep learning tech on language with Neural Machine Translation

*Posted Sep 27, 2016 by **Devin Coldewey**, Contributor*



Systran launches neural machine translation engine in 30 languages

Language barriers represent one of the biggest challenges to develop business strategies among global markets. Now, thanks to advances in artificial intelligence and machine translation, these barriers are being broken down.

By Eileen Brown for Social Business | November 24, 2016 -- 13:49 GMT (13:49 GMT) | Topic: Artificial Intelligence

Siri and Alexa Are Fighting to Be Your Hotel Butler

By **Hui-yong Yu** and **Spencer Soper**
March 22, 2017, 9:00 AM GMT *Updated on* March 22, 2017, 2:13 PM GMT

- Hotels are new frontier for voice-command technologies
- Wynn Las Vegas was first to install Alexa devices in December

AlphaGo Beats Go Human Champ: Godfather Of Deep Learning Tells Us Do Not Be Afraid Of AI

21 March 2016, 10:16 am EDT **By Aaron Mamiit Tech Times**

Last week, Google's artificial intelligence program

Last week, Google's artificial intelligence program AlphaGo dominated its match with South Korean world Go champion Lee Sedol, winning with a 4-1 score.

The achievement stunned artificial intelligence experts, who previously thought that Google's computer program would need at least 10 more years before developing enough to be able to beat a human world champion.

A robot wrote this entire article. Are you scared yet, human?

We asked GPT-3, OpenAI's powerful new language generator, to write an essay for us from scratch. The assignment? To convince us robots come in peace

- For more about GPT-3 and how this essay was written and edited, please read our editor's note below

Why Now?

Why does deep learning work now, but not 20 years ago?

Many of the core ideas were there, after all.

But now we have:

- more data
- more computing power
- better software engineering
- a few algorithmic innovations (many layers, ReLUs, better initialization and learning rates, dropout, LSTMs, convolutional nets)

"But It's Non-Convex"

Why does gradient-based optimization work at all in neural nets despite the non-convexity?

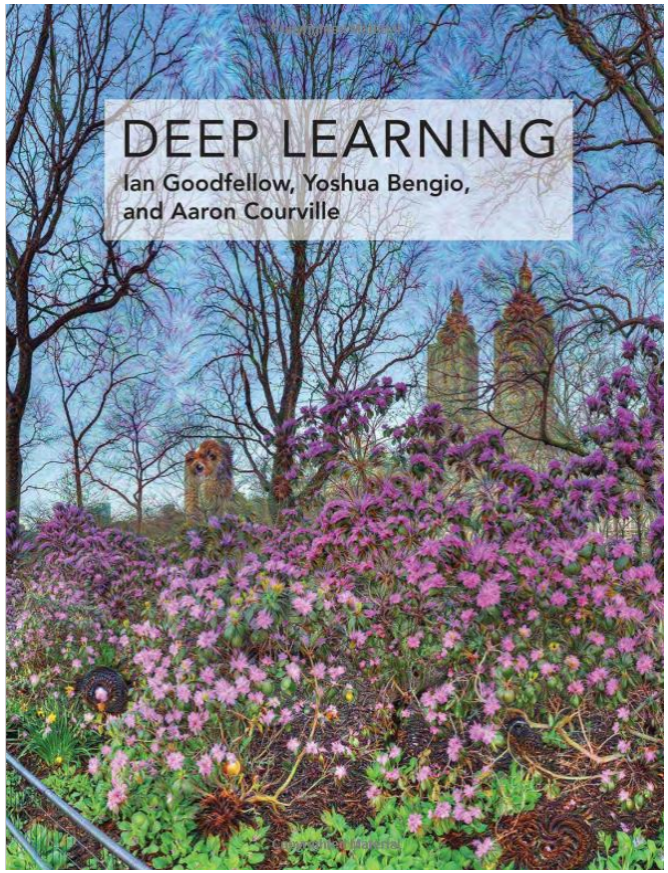
One possible, partial answer:

- there are generally many hidden units
- there are many ways a neural net can approximately implement the desired input-output relationship
- we only need to find one

Recommended Books

Main book:

- **Deep Learning.** Ian Goodfellow, Yoshua Bengio, and Aaron Courville. MIT Press, 2016. Chapters available at http://deeplearningbook.org



Recommended Books

Secondary books:

- **Machine Learning: a Probabilistic Perspective.** Kevin P. Murphy. MIT Press, 2013.
- **Introduction to Natural Language Processing.** Jacob Eisenstein. MIT Press. 2019.
- **Linguistic Structured Prediction.** Noah A. Smith. Morgan & Claypool Synthesis Lectures on Human Language Technologies. 2011.

Tentative Syllabus

Sep 23	Introduction and Course Description
Sep 30	Linear Classifiers
Oct 7	Feedforward Neural Networks
Oct 14	Representation Learning and Convolutional Networks
Oct 21	Neural Network Toolkits (Pytorch)
Oct 28	Linear Sequence Models
Nov 4	Recurrent Neural Networks
Nov 11	Structured Prediction and Graphical Models
Nov 18	Sequence-to-Sequence Learning
Nov 25	Attention Mechanisms and Explainability
Dec 2	Deep Reinforcement Learning
Dec 9	Deep Generative Models (VAEs, GANs)
January	Final Projects

Outline

What This Class Is About

- Introduction to deep learning
- Introduction to structured prediction

- **Goal:** after finishing this class, you should be able to:
  - Understand how deep learning works without magic
  - Understand the intuition behind deep structured learning models
  - Apply the learned techniques on a practical problem (NLP, vision, ...)

- **Target audience:**
  - MSc/PhD students with basic background in ML and good programming skills

What This Class Is Not About

It's **not** about:

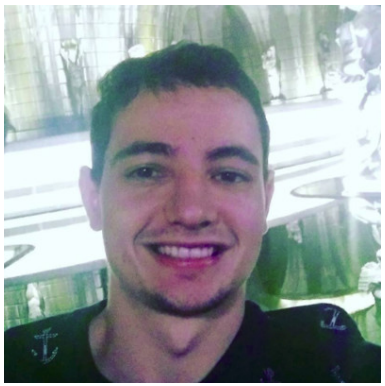
- Just playing with a deep learning toolkit without learning the fundamental concepts
- Introduction to ML (see Mário Figueiredo's Statistical Learning course and Jorge Marques' Estimation and Classification course)
- Optimization (check João Xavier's Non-Linear Optimization course)
- Natural Language Processing
- ...

Prerequisites

- Calculus and basic linear algebra
- Basic probability theory
- Basic knowledge of machine learning
- Programming (Python & PyTorch preferred)
- Helpful: basic optimization

Course Information

- **Instructor:** André Martins
- **TA:** Marcos Treviso
- **Location:** Fully remote (Zoom link in Piazza)
- **Schedule:** Wednesdays 14:30–17:30 (tentative)
- **Communication:** piazza.com/tecnico.ulisboa.pt/fall2020/pdeecdsl



Grading

- 3 homework assignments: 60%
  - Theoretical questions & implementation
  - Late days: 10% penalization each late day
- Final project (in groups of 2–3): 40%
  - Final class presentations

Final Project

- **Possible idea:** apply a deep learning technique to a structured problem relevant to your research (NLP, vision, robotics, ...)
- Otherwise, pick a project from a list of suggestions
- Must be finished this semester
- Four evaluation stages: project proposal (10%), midterm report (10%), final report (10%, conference paper format), class presentation (10%)
- List of project suggestions will be made available soon

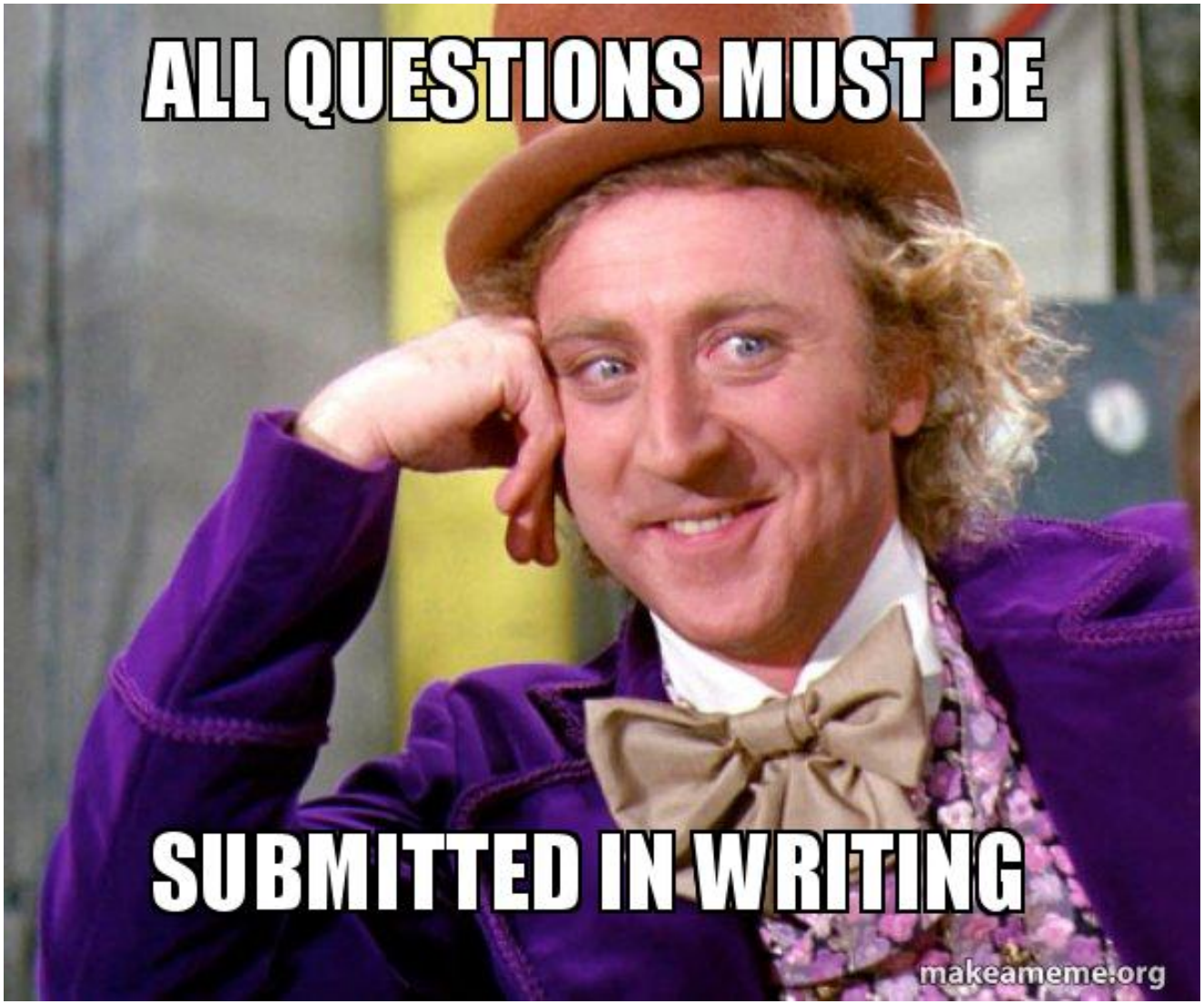
Collaboration Policy

- Assignments are individual
- Students may discuss the questions, as long as they write their own answers and their own code
- If this happens, acknowledge with whom you collaborate!
- Zero tolerance on plagiarism!!
- Always credit your sources!!!

Caveat

- This is the third year I'm teaching this class
- ... which means you're the third batch of students taking it :)
- Constructive feedback will be highly appreciated (and encouraged!)

Questions?



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Quick Background Recap

Slide credits: Prof. Mário Figueiredo (taken from his LxMLS class)

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Linear Algebra

- Linear algebra provides (among many other things) a compact way of representing, studying, and solving linear systems of equations

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- Example: the system

$$\begin{aligned} 4\,x_1 - 5\,x_2 &= -13 \\ -2\,x_1 + 3\,x_2 &= 9 \end{aligned}$$

can be written compactly as $\boxed{Ax = b}$, where

$$A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}, \; b = \begin{bmatrix} -13 \\ 9 \end{bmatrix},$$

and can be solved as

$$x = A^{-1}b = \begin{bmatrix} 1.5 & 2.5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -13 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Notation: Matrices and Vectors

- $A \in \mathbb{R}^{m \times n}$ is a matrix with m rows and n columns.

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix}.$$

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Matrix Transpose and Products

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- Outer product between vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$: $x\,y^T \in \mathbb{R}^{n \times m}$, where $(x\,y^T)_{i,j} = x_i\,y_j$.

Properties of Matrix Products and Transposes

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- In general, matrix product is not commutative: $AB \neq BA$.
- Transpose of product: $(AB)^T = B^T A^T$.
- Transpose of sum: $(A + B)^T = A^T + B^T$.

Norms

- The norm of a vector is (informally) its "magnitude." Euclidean norm:

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- Notable case: the ℓ_∞ norm, $\|x\|_\infty = \max\{|x_1|, ..., |x_n|\}$.
- Notable case: the ℓ_0 "norm" (not): $\|x\|_0 = |\{i : x_i \neq 0\}|$.

Special Matrices

- The identity matrix $I \in \mathbb{R}^{n \times n}$ is a square matrix such that

$$I_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

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- Upper triangular matrix: $(j < i) \Rightarrow A_{i,j} = 0$.
- Lower triangular matrix: $(j > i) \Rightarrow A_{i,j} = 0$.

Eigenvalues, eigenvectors, determinant, trace

- A vector $x \in \mathbb{R}^n$ is an eigenvector of matrix $A \in \mathbb{R}^{n \times n}$ if

$$A x = \lambda x,$$

where $\lambda \in \mathbb{R}$ is the corresponding eigenvalue.

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- Properties: $|AB| = |A||B|$,

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Eigenvalues, eigenvectors, determinant, trace

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- There are many algorithms to compute A^{-1}; general case, computational cost $O(n^3)$.

Quadratic Forms and Positive (Semi-)Definite Matrices

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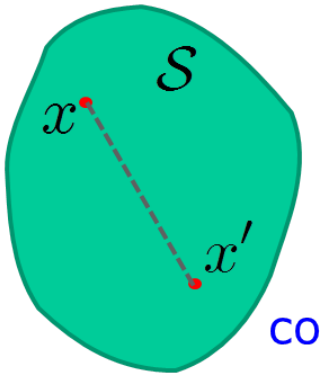
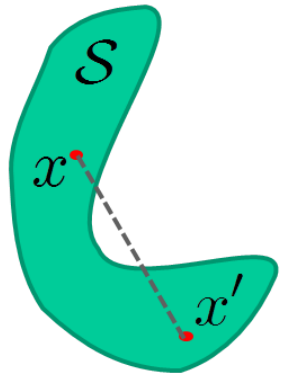
Convex and strictly convex sets

$\mathcal{S}$ is convex if $x, x' \in \mathcal{S} \Rightarrow \forall \lambda \in [0,1], \quad \lambda x + (1-\lambda)x' \in \mathcal{S}$



convex



non-convex

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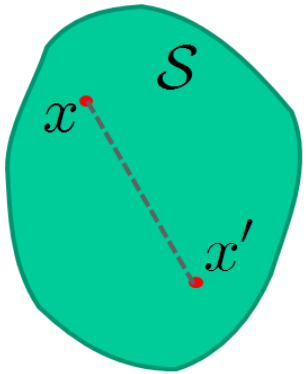

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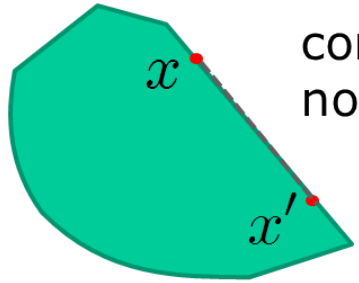

convex, but not strictly

Convex and strictly convex functions

Extended real valued function: $f : \mathbb{R}^N \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$

Domain of a function: $\mathrm{dom}(f) = \{x : f(x) \neq +\infty\}$

f is a convex function if

$$\forall \lambda \in [0,1], x, x' \in \mathrm{dom}(f) \quad f(\lambda x + (1-\lambda)x') \leq \lambda f(x) + (1-\lambda) f(x')$$

f is a strictly convex function if

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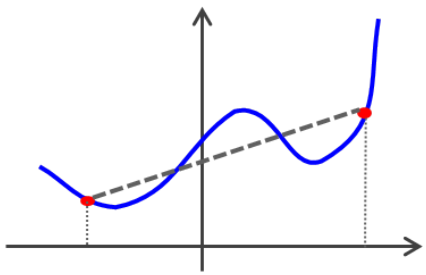

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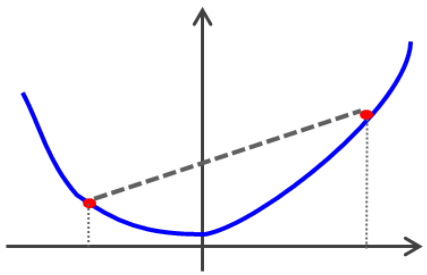

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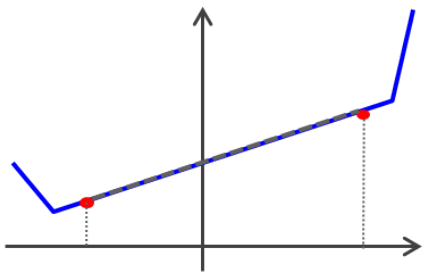

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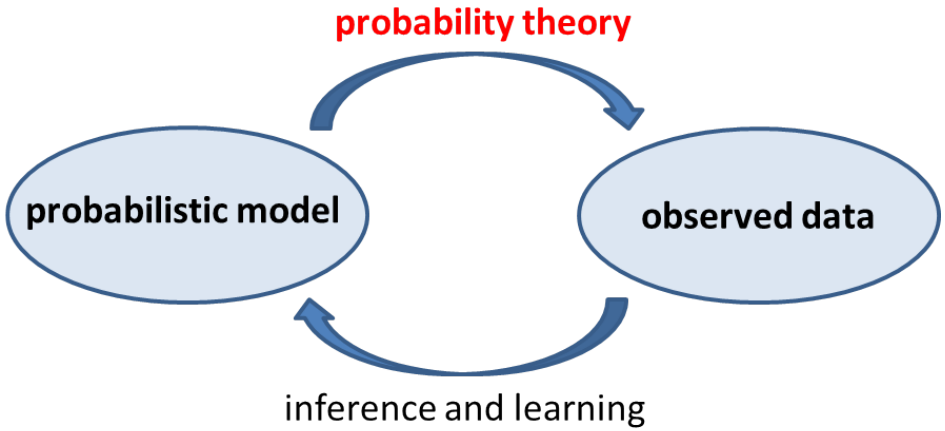
Outline

Probability theory



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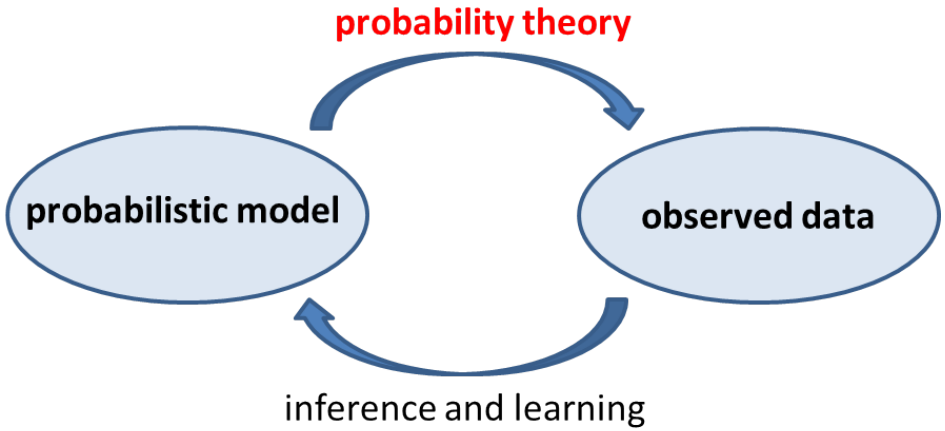

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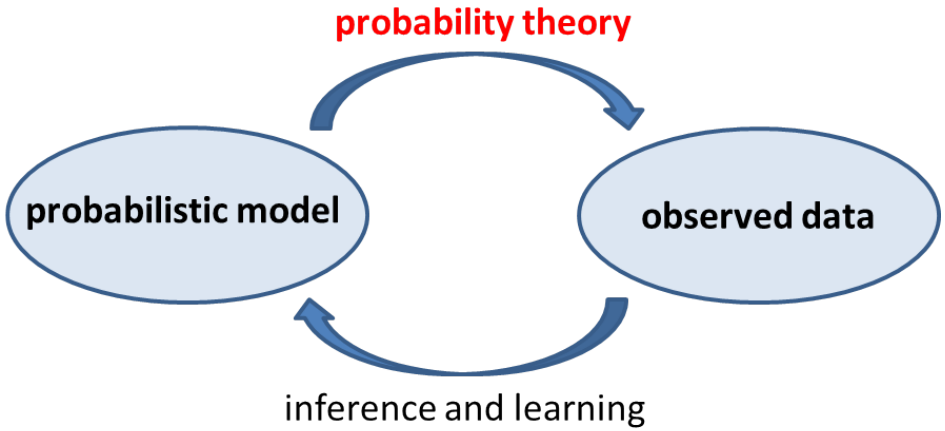

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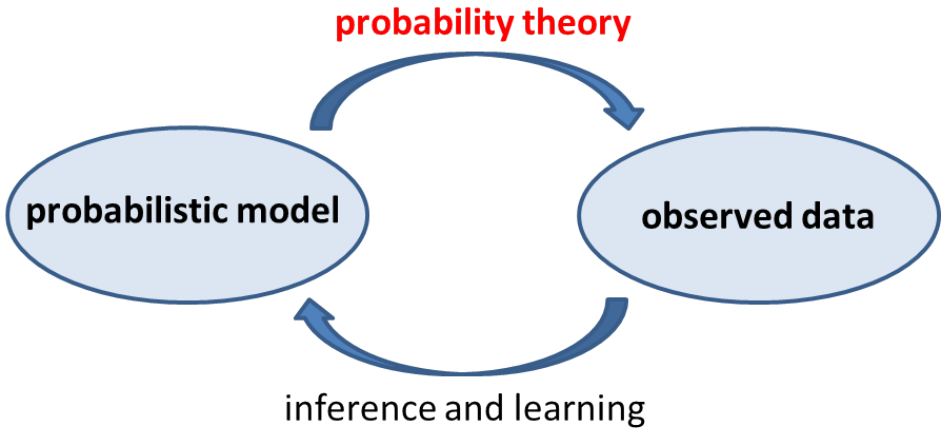

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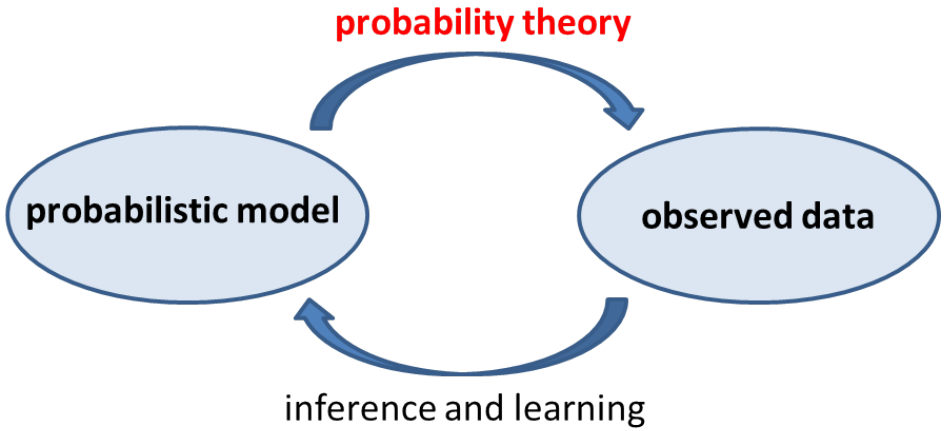

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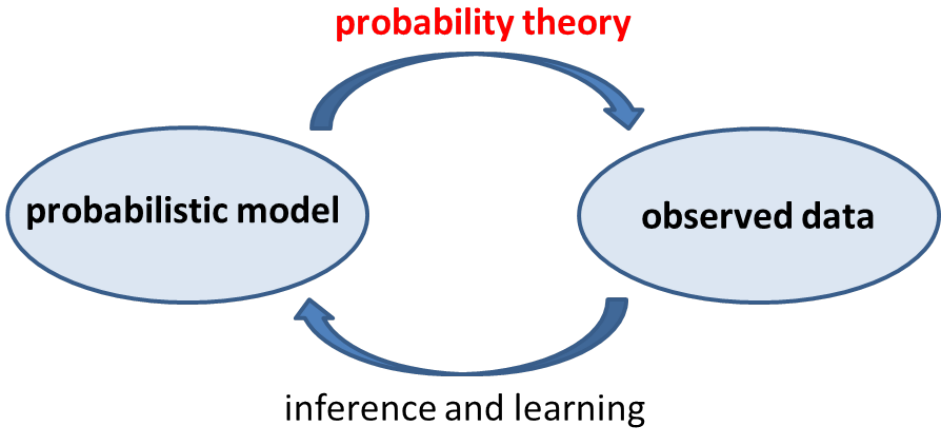

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- Natural tool to model uncertainty, information, knowledge, belief, ...
- ...thus also learning, decision making, inference, ...

What is probability?

- Classical definition: $\mathbb{P}(A) = \dfrac{N_A}{N}$

  ...with N mutually exclusive equally likely outcomes,
  N_A of which result in the occurrence of event A. *Laplace, 1814*

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- Subjective probability: $\mathbb{P}(A)$ is a degree of belief. *de Finetti, 1930s*

  ...gives meaning to $\mathbb{P}(\text{“Tomorrow it will rain”})$.

Key concepts: Sample space and events

- Sample space $\mathcal{X}$ = set of possible outcomes of a random experiment.

  Examples:
  - Tossing two coins: $\mathcal{X} = \{HH, TH, HT, TT\}$
  - Roulette: $\mathcal{X} = \{1, 2, ..., 36\}$
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- An event A is a subset of $\mathcal{X}$: $A \subseteq \mathcal{X}$.

  Examples:
  - "exactly one H in 2-coin toss": $A = \{TH, HT\} \subset \{HH, TH, HT, TT\}$.
  - "odd number in the roulette": $B = \{1, 3, ..., 35\} \subset \{1, 2, ..., 36\}$.
  - "drawn a $\heartsuit$ card": $C = \{A\heartsuit, 2\heartsuit, ..., K\heartsuit\} \subset \{A\clubsuit, ..., K\diamondsuit\}$

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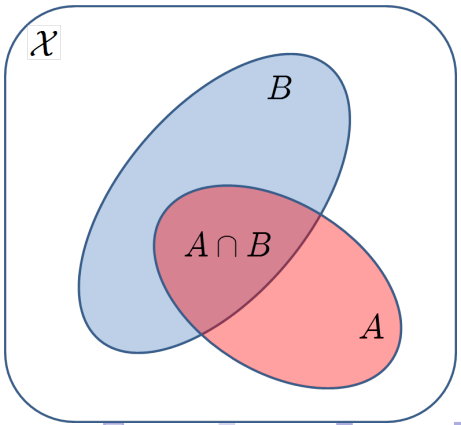
- From these axioms, many results can be derived. Examples:

- $\mathbb{P}(\emptyset) = 0$
- $C \subset D \Rightarrow \mathbb{P}(C) \leq \mathbb{P}(D)$
- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
- $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$ (union bound)



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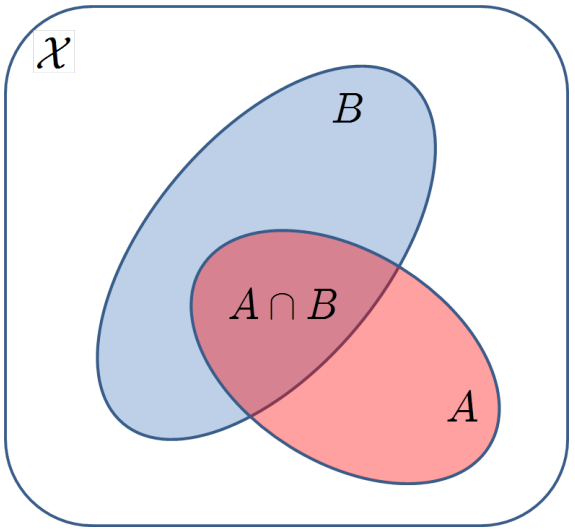

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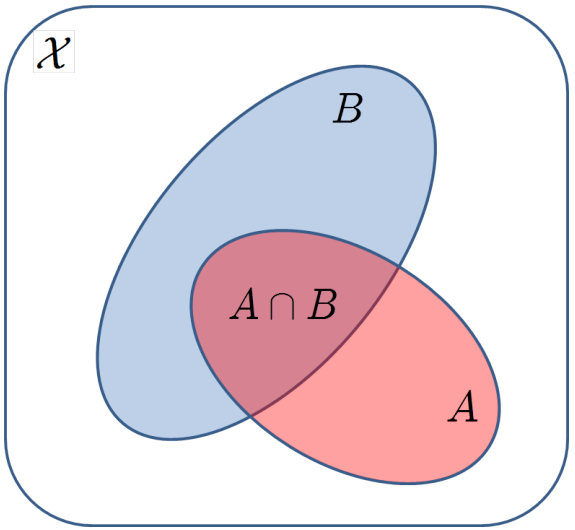

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- Example: $\mathcal{X} =$ "52 cards", $A = \{3\heartsuit, 3\clubsuit, 3\diamondsuit, 3\spadesuit\}$, and $B = \{A\heartsuit, 2\heartsuit, ..., K\heartsuit\}$; then, $\mathbb{P}(A) = 1/13$, $\mathbb{P}(B) = 1/4$

$$\mathbb{P}(A \cap B) \quad = \quad \mathbb{P}(\{3\heartsuit\}) = \frac{1}{52}$$

Conditional Probability and Independence

- If $\mathbb{P}(B) > 0$, $\quad \mathbb{P}(A|B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$
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- Law of total probability: if $A_1, ..., A_n$ are a partition of $\mathcal{X}$

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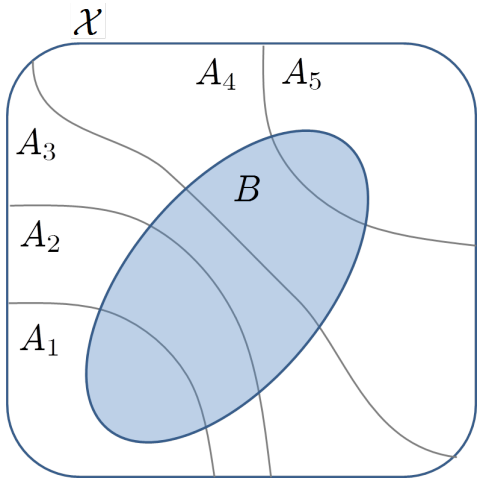

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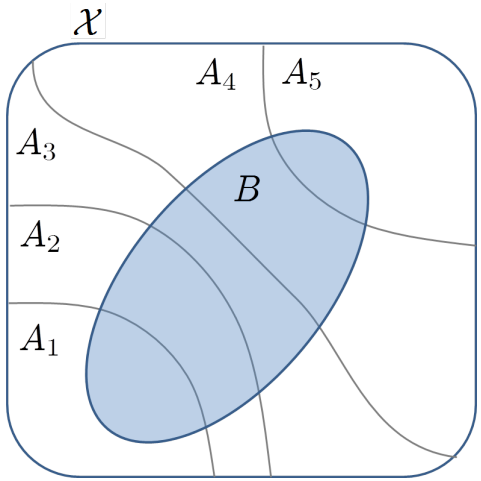

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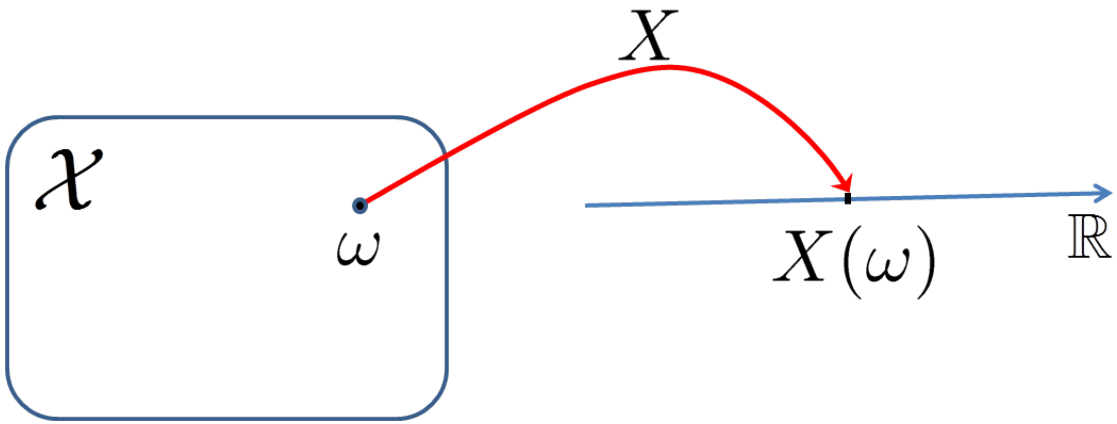
Random Variables

- A (real) random variable (RV) is a function: $X : \mathcal{X} \to \mathbb{R}$



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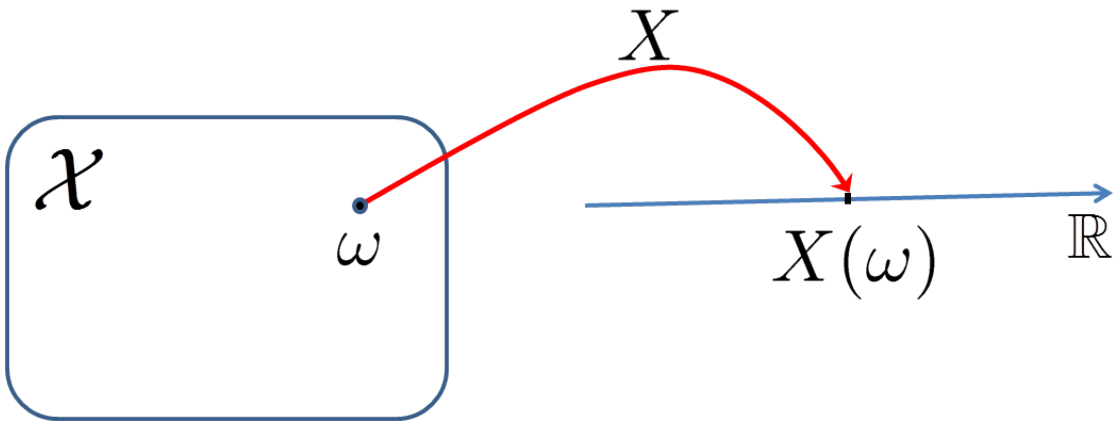

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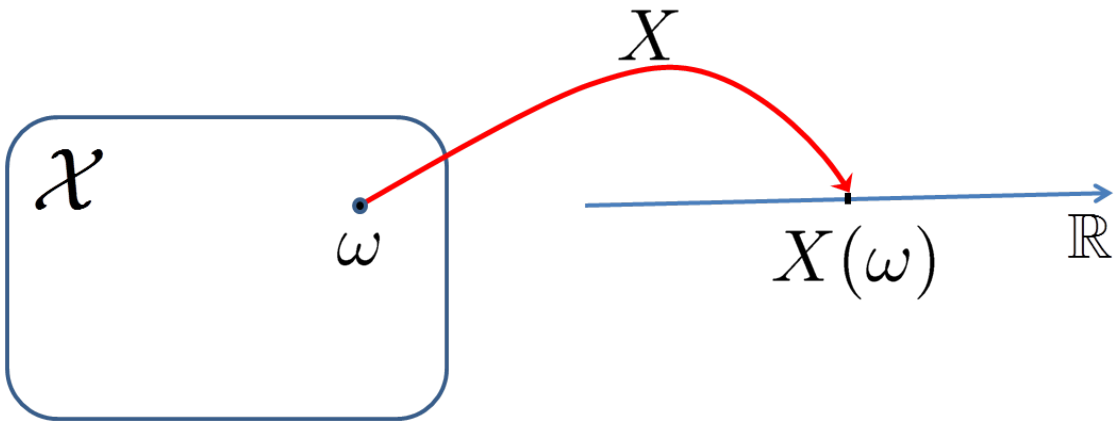

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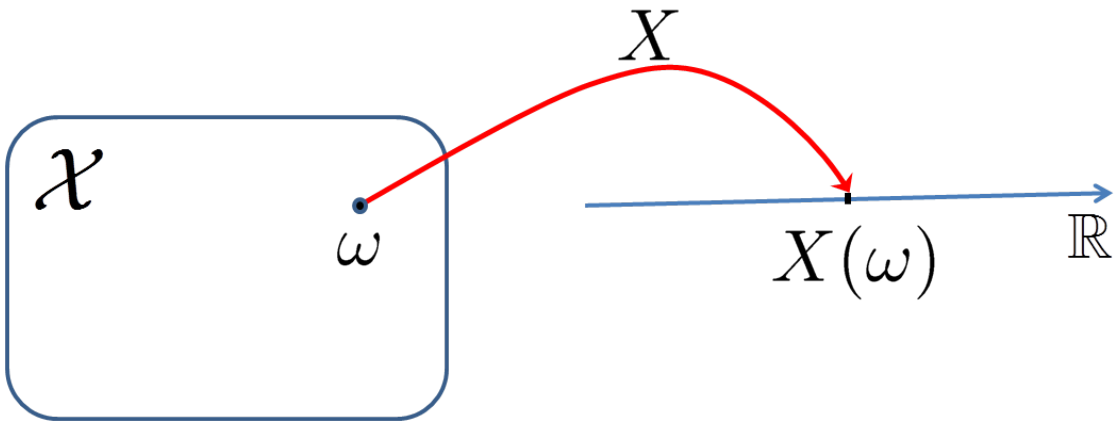

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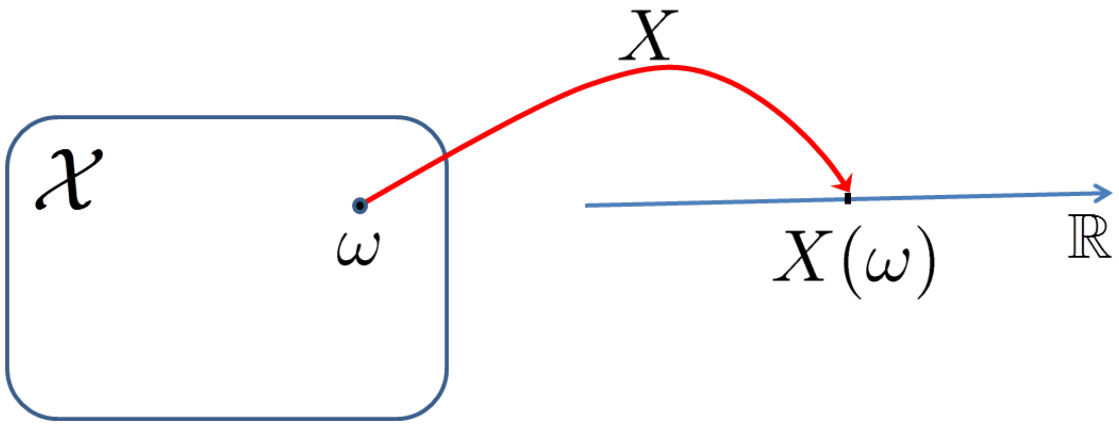

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Random Variables: Distribution Function

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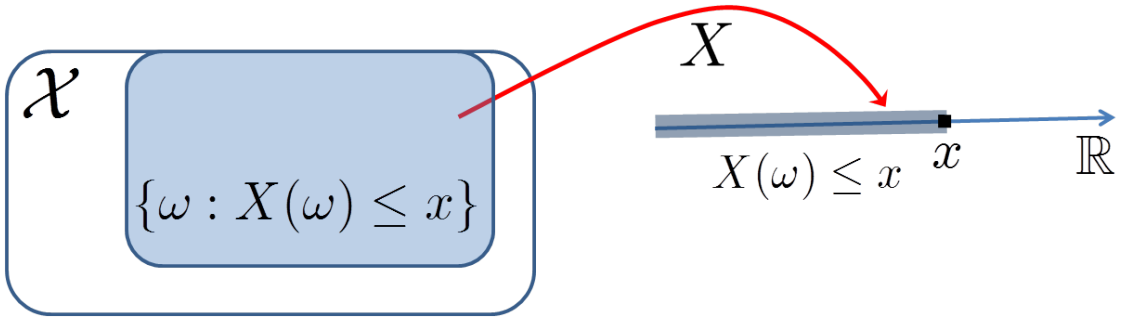

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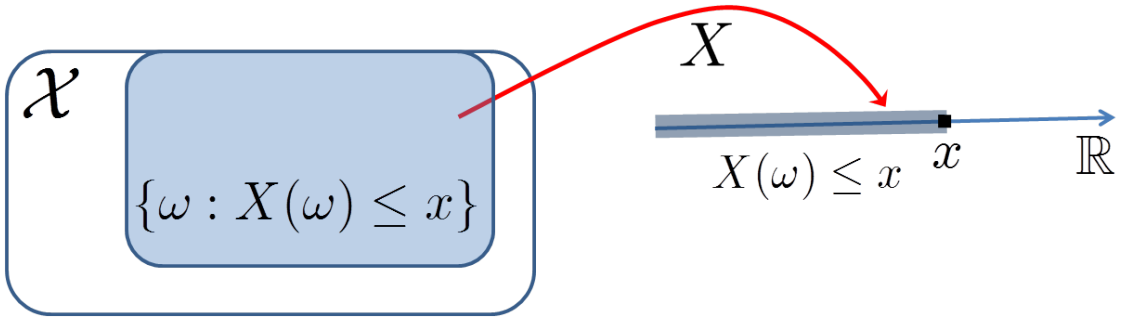

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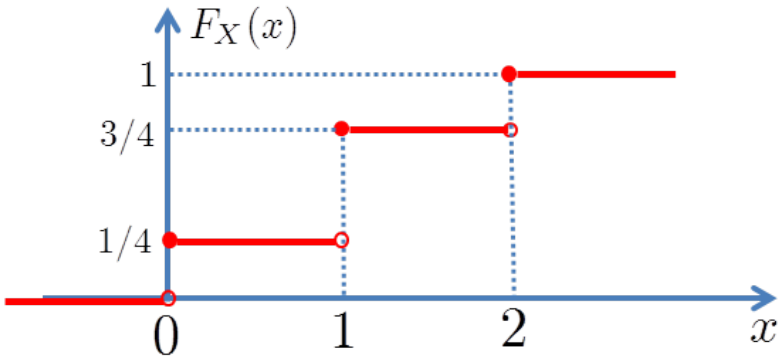

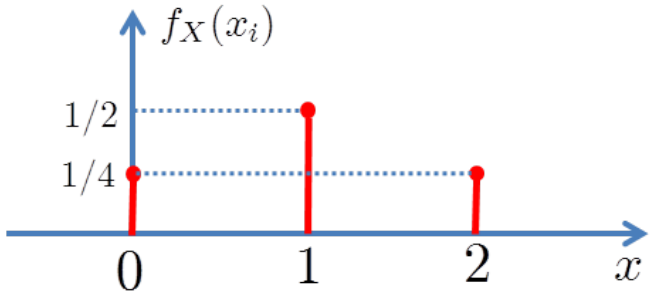

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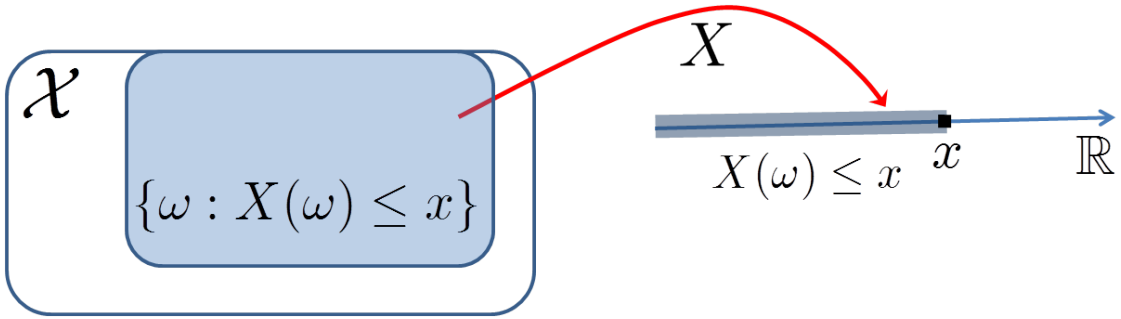

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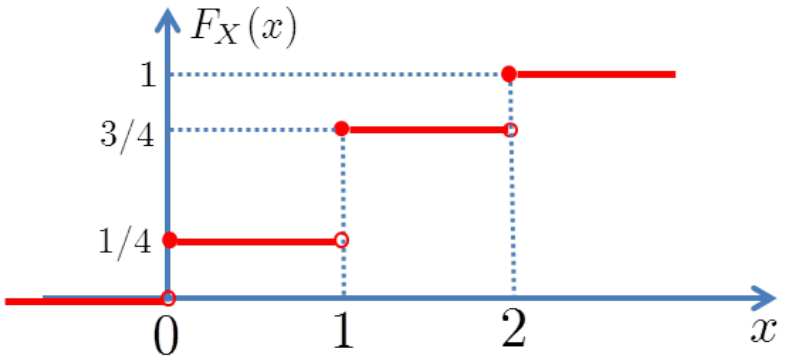

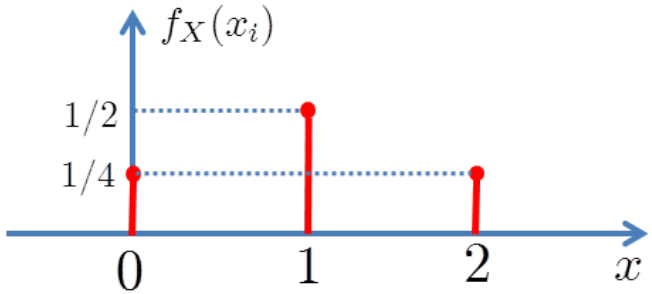

- Probability mass function (discrete RV): $f_X(x) = \mathbb{P}(X = x)$,

$$F_X(x) = \sum_{x_i \leq x} f_X(x_i).$$

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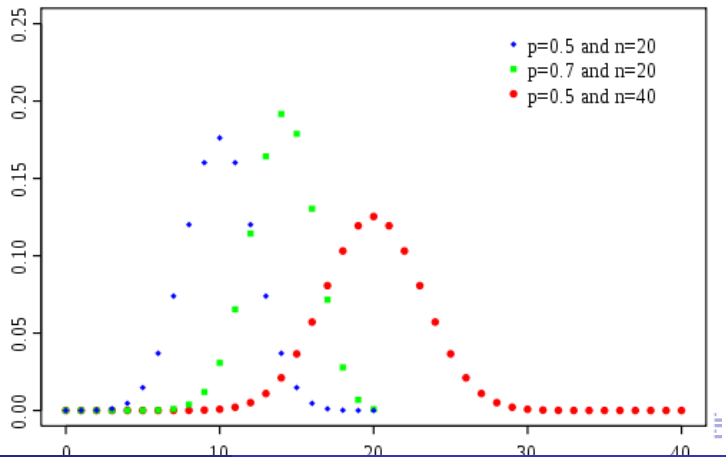
Binomial coefficients ("n choose x"):

$$\binom{n}{x} = \frac{n!}{(n-x)!\, x!}$$



More Important Discrete Random Variables

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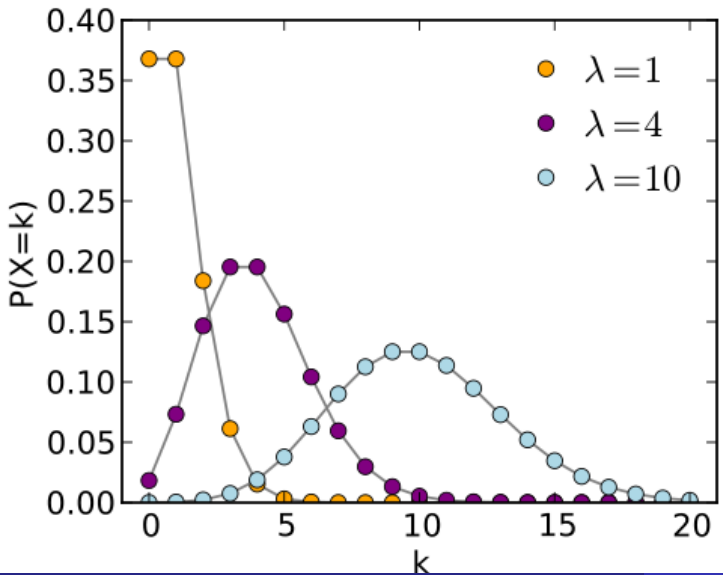
- Poisson(λ): $X \in \mathbb{N} \cup \{0\}$, pmf $f_X(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$

  Notice that $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$, thus $\sum_{x=0}^{\infty} f_X(x) = 1$.

  "...probability of the number of independent occurrences in a fixed (time/space) interval if these occurrences have known average rate"



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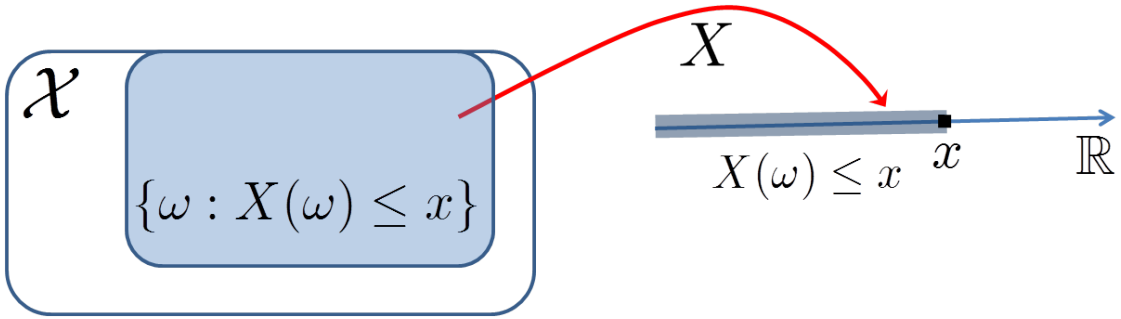

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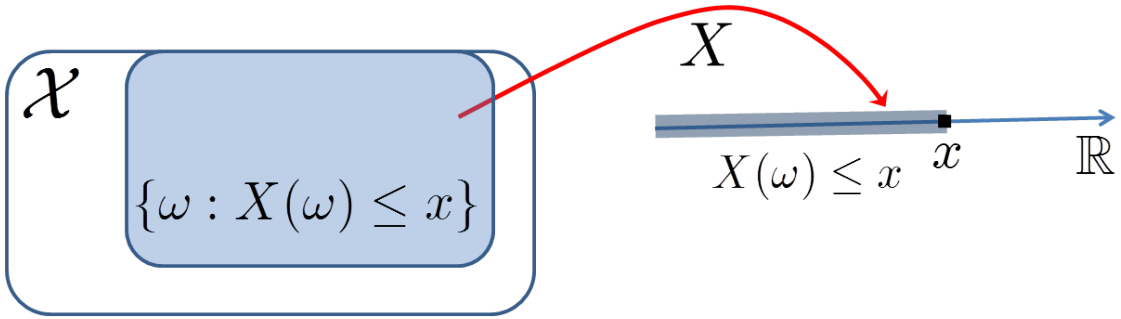
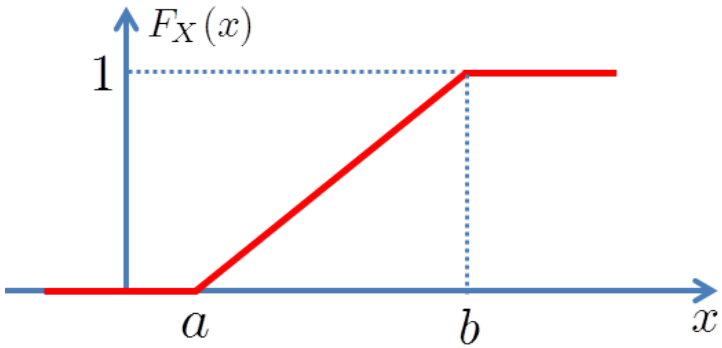
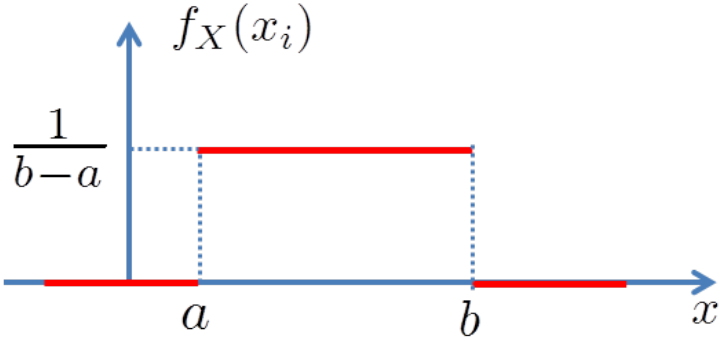


- Example: continuous RV with uniform distribution on $[a, b]$.





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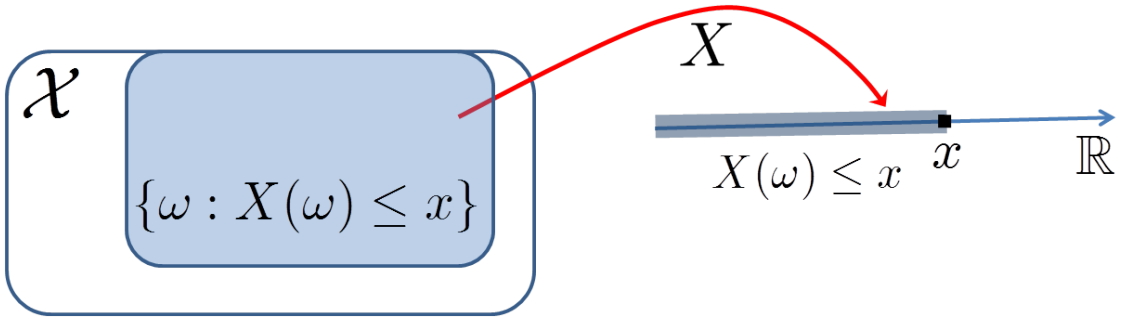

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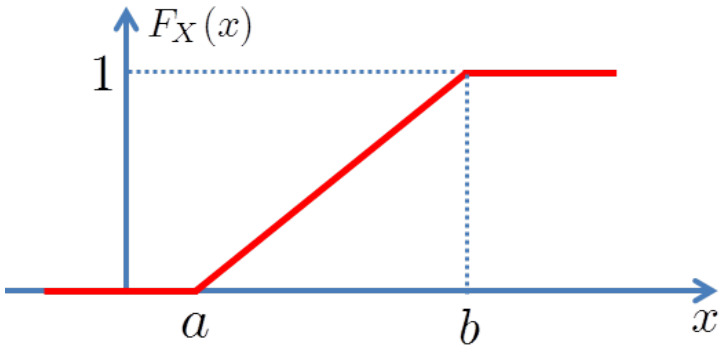

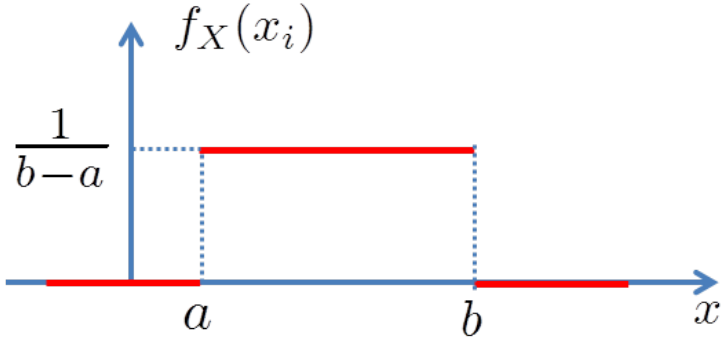

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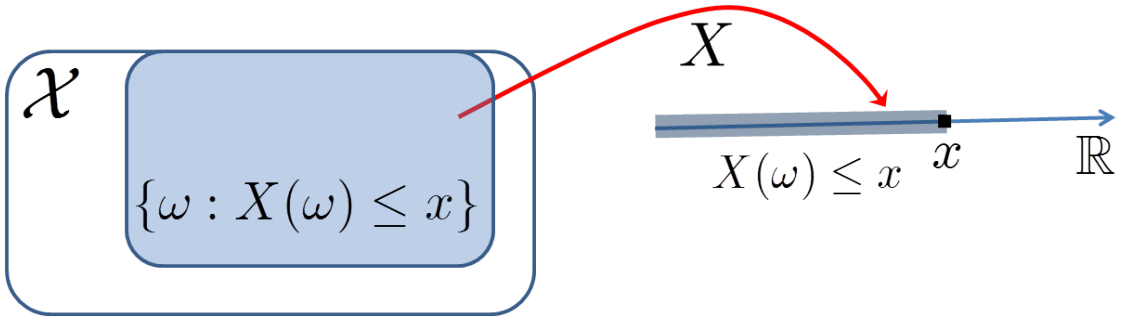

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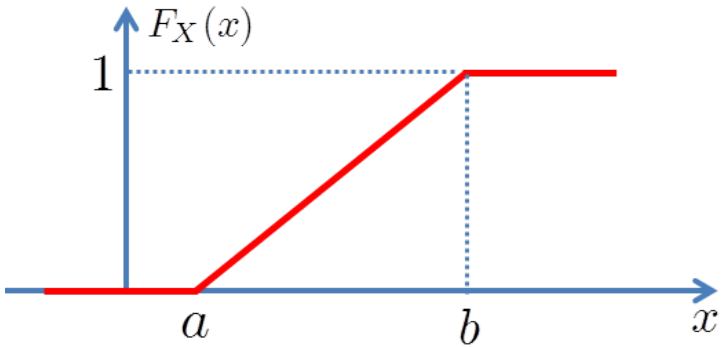

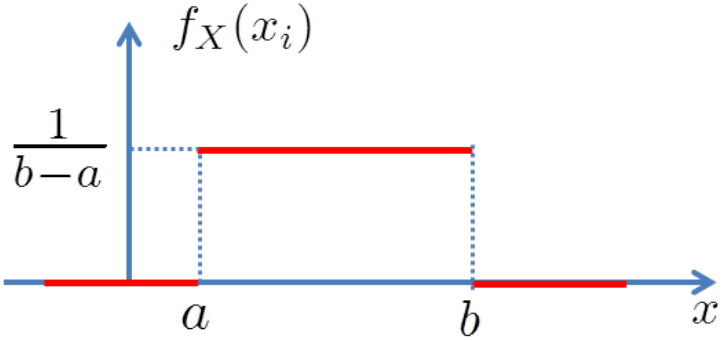

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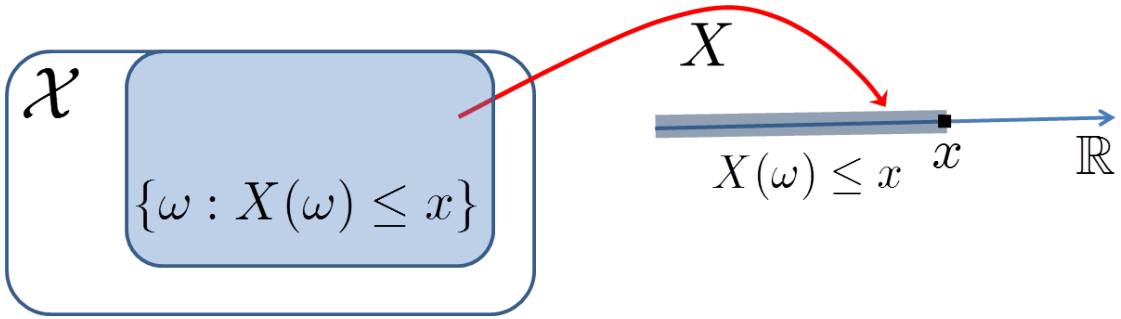

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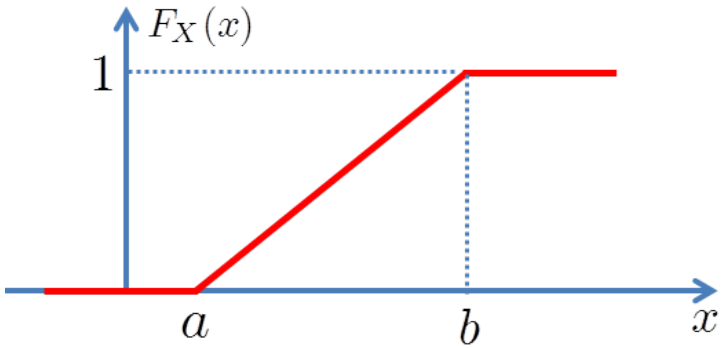

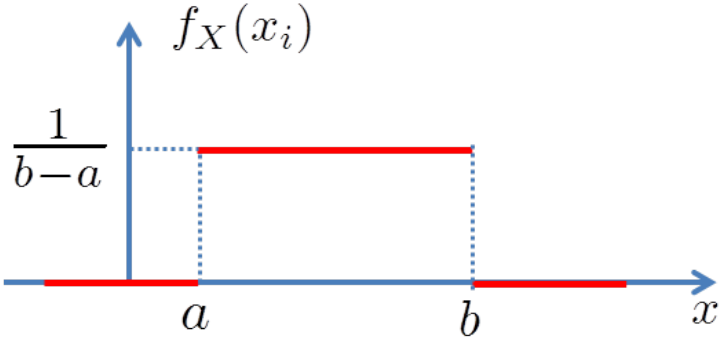

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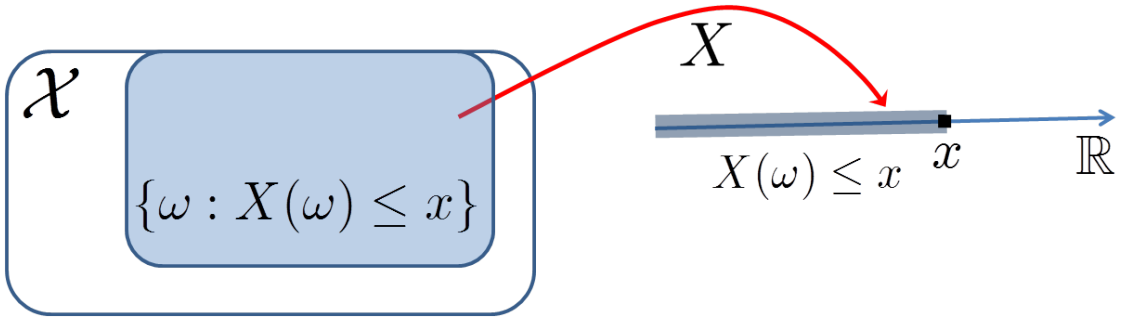

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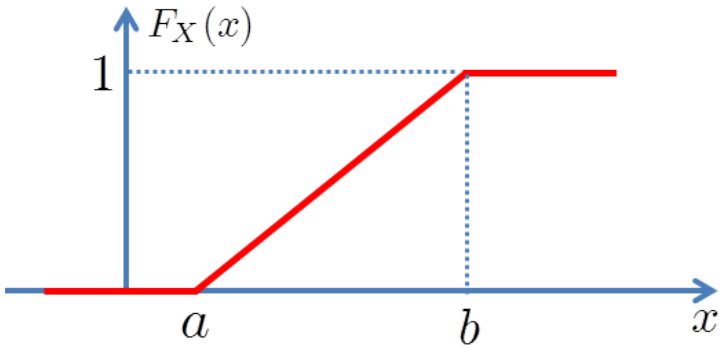

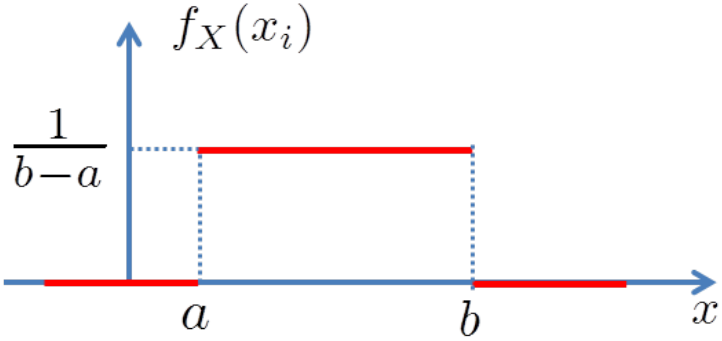

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- Uniform: $f_X(x) = \text{Uniform}(x; a, b) = \begin{cases} \frac{1}{b-a} & \Leftarrow \quad x \in [a,\ b] \\ 0 & \Leftarrow \quad x \notin [a,\ b] \end{cases}$

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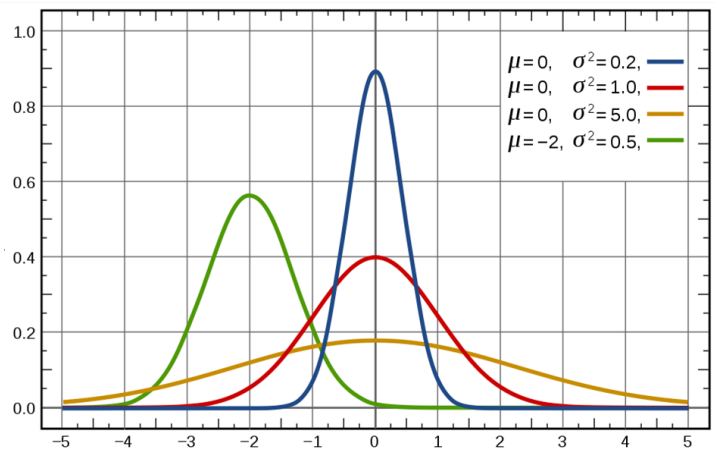
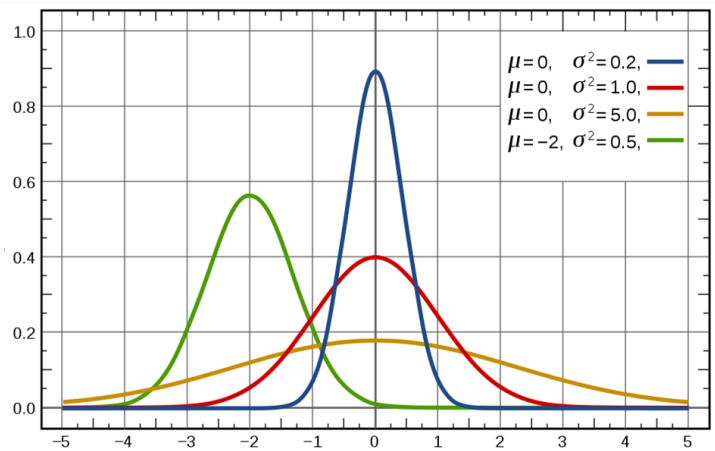


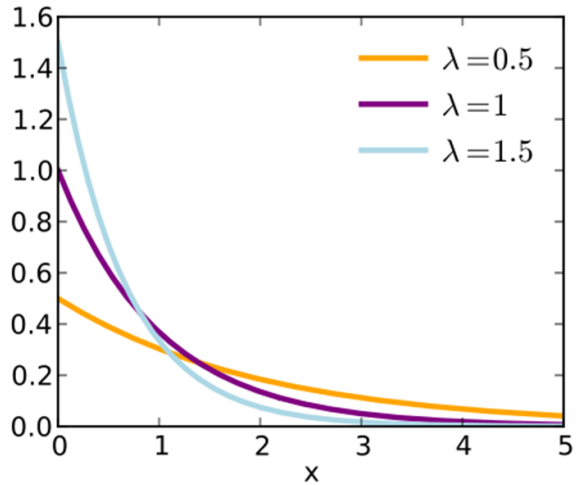
Important Continuous Random Variables

- Uniform: $f_X(x) = \text{Uniform}(x; a, b) = \begin{cases} \frac{1}{b-a} & \Leftarrow \quad x \in [a, b] \\ 0 & \Leftarrow \quad x \notin [a, b] \end{cases}$

  (previous slide).

- Gaussian: $f_X(x) = \mathcal{N}(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\,\pi\,\sigma^2}} e^{-\frac{(x-\mu)^2}{2\,\sigma^2}}$





- Exponential: $f_X(x) = \text{Exp}(x; \lambda) = \begin{cases} \lambda e^{-\lambda\,x} & \Leftarrow \quad x \geq 0 \\ 0 & \Leftarrow \quad x < 0 \end{cases}$

Expectation of Random Variables

- Expectation: $\mathbb{E}(X) = \begin{cases} \sum_i x_i \, f_X(x_i) & X \in \{x_1, ..., x_K\} \subset \mathbb{R} \\ \int_{-\infty}^{\infty} x \, f_X(x) \, dx & X \text{ continuous} \end{cases}$

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- Linearity of expectation:
  $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$; $\mathbb{E}(\alpha \, X) = \alpha \mathbb{E}(X)$, $\alpha \in \mathbb{R}$

Expectation of Functions of Random Variables

- $$\mathbb{E}(g(X)) = \begin{cases} \sum_i g(x_i) f_X(x_i) & X \text{ discrete, } g(x_i) \in \mathbb{R} \\ \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx & X \text{ continuous} \end{cases}$$

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- Probability as expectation of indicator, $\mathbf{1}_A(x) = \begin{cases} 1 & \Leftarrow \quad x \in A \\ 0 & \Leftarrow \quad x \notin A \end{cases}$

$$\mathbb{P}(X \in A) = \int_A f_X(x)\, dx = \int \mathbf{1}_A(x)\, f_X(x)\, dx = \mathbb{E}(\mathbf{1}_A(X))$$

Two (or More) Random Variables

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Conditionals and Bayes' Theorem

- Conditional pmf (discrete RVs):

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- Also valid in the mixed case (*e.g.*, X continuous, Y discrete).

Joint, Marginal, and Conditional Probabilities: An Example

- A pair of binary variables $X, Y \in \{0, 1\}$, with joint pmf:

$f_{X,Y}(x,y)$	$Y=0$	$Y=1$
$X=0$	1/5	2/5
$X=1$	1/10	3/10

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$f_{X\|Y}(x\|y)$	$Y=0$	$Y=1$
$X=0$	2/3	4/7
$X=1$	1/3	3/7

$f_{Y\|X}(y\|x)$	$Y=0$	$Y=1$
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$X=1$	1/4	3/4

An Important Multivariate RV: Multinomial

- Multinomial: $X = (X_1, ..., X_K)$, $X_i \in \{0, ..., n\}$, such that $\sum_i X_i = n$,

$$f_X(x_1, ..., x_K) = \begin{cases} \binom{n}{x_1\ x_2 \cdots x_K} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_K} & \Leftarrow \sum_i x_i = n \\ 0 & \Leftarrow \sum_i x_i \neq n \end{cases}$$

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- Generalizes the binomial from binary to K-classes.

- Example: tossing n independent fair dice, $p_1 = \cdots = p_6 = 1/6$.
  x_i = number of outcomes with i dots. Of course, $\sum_i x_i = n$.

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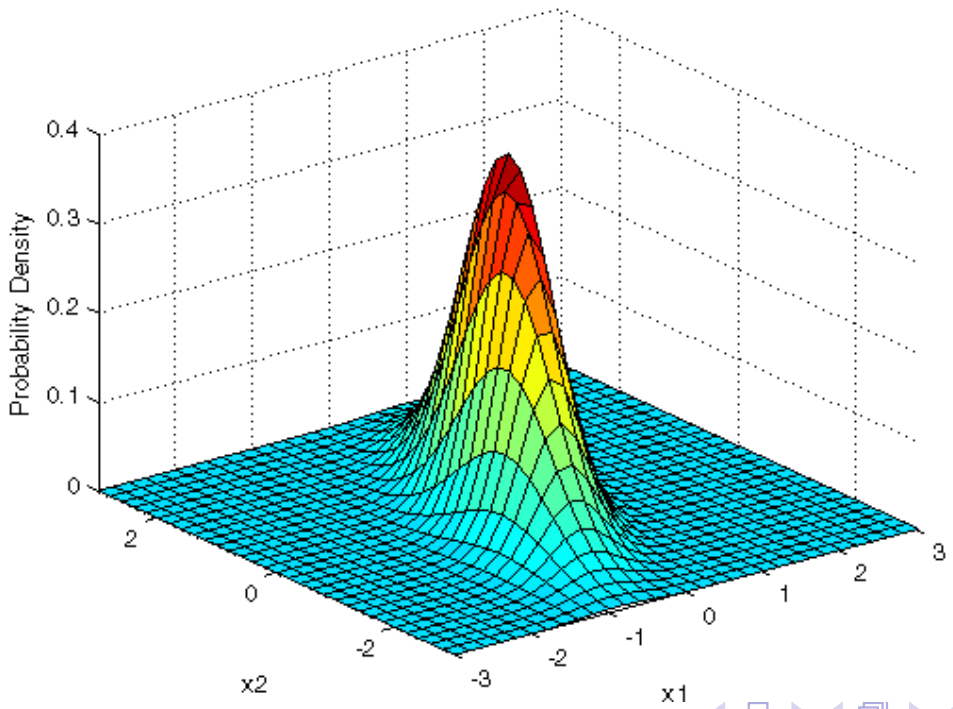

Covariance, Correlation, and all that...

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- Covariance of Gaussian RV, $f_X(x) = \mathcal{N}(x; \mu, C) \;\Rightarrow\; \text{cov}(X) = C$

More on Expectations and Covariances

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- ...thus, if $Z_n = \dfrac{Y_n - \mu}{\sigma}$

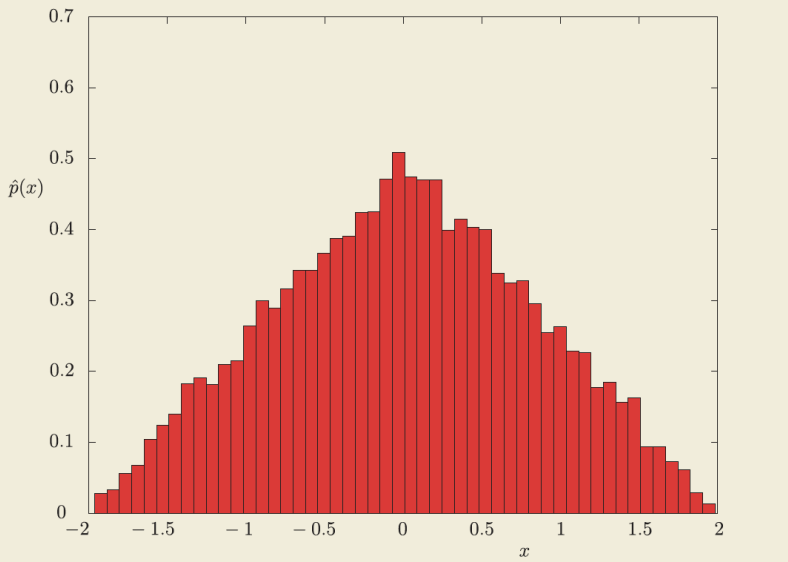
$$\mathbb{E}[Z_n] = 0 \qquad \text{var}(Z_n) = 1$$

- Central limit theorem (CLT): under some mild conditions on $X_1, ..., X_n$

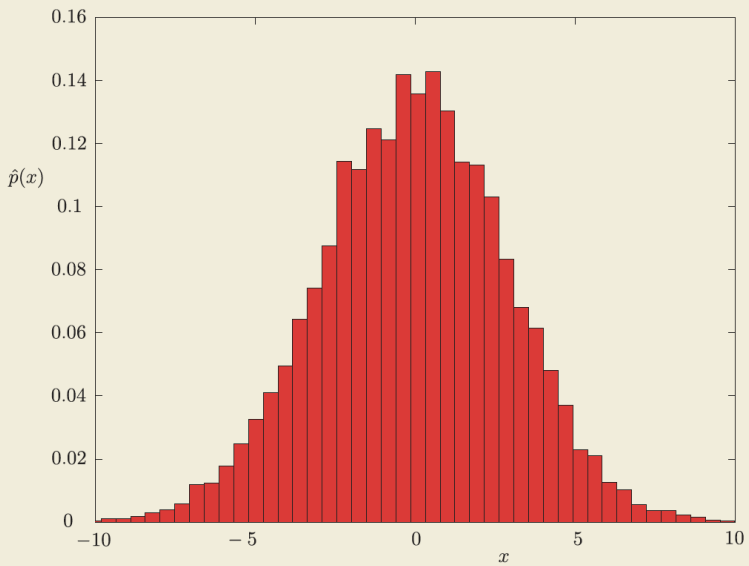
$$\lim_{n \to \infty} Z_n \sim \mathcal{N}(0, 1)$$

Central Limit Theorem

Illustration



Sum of two i.i.d variables from a uniform in [-1,1]



Sum of twenty five i.i.d r.v from a uniform in [-1,1]

Important Inequalities

- Markov's ineqality: if $X \geq 0$ is an RV with expectation $\mathbb{E}(X)$, then

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- Chebyshev's inequality: $\mu = \mathbb{E}(Y)$ and $\sigma^2 = \text{var}(Y)$, then

$$\mathbb{P}(|Y - \mu| \geq s) \leq \frac{\sigma^2}{s^2}$$

...simple corollary of Markov's inequality, with $X = |Y - \mu|^2$, $t = s^2$

Other Important Inequalities: Cauchy-Schwartz

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- Implication for correlation:

$$\text{corr}(X, Y) = \rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}} \in [-1, 1]$$

Other Important Inequalities: Jensen

- Recall that a real function g is convex if, for any x, y, and $\alpha \in [0, 1]$

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Examples: $\mathbb{E}(X)^2 \leq \mathbb{E}(X^2) \Rightarrow \text{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \geq 0.$

$$\mathbb{E}(\log X) \leq \log \mathbb{E}(X), \quad \text{for } X \text{ a positive RV.}$$

Entropy and all that...

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Kullback-Leibler divergence

Kullback-Leibler divergence (KLD) between two pmf:

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Mutual information

Mutual information (MI) between two random variables:

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MI is a measure of dependency between two random variables

Recommended Reading

- K. Murphy, "Machine Learning: A Probabilistic Perspective", MIT Press, 2012.
- L. Wasserman, "All of Statistics: A Concise Course in Statistical Inference", Springer, 2004.

Outline

Minimizing a function

- We are given a function $f : \mathbb{R}^n \to \mathbb{R}$.
- Goal: find x^* that minimizes $f : \mathbb{R}^n \to \mathbb{R}$.
- Global minimum: for any $x \in \mathbb{R}^n$, $f(x^*) \leq f(x)$.
- Local minimum: for any $\|x - x^*\| \leq \delta \Rightarrow f(x^*) \leq f(x)$.

Are these global minima ?

- No, (local minima, saddle points, . . .)

Iterative descent methods

Goal: find the minimum/minimizer of $f : \mathbb{R}^d \to \mathbb{R}$

- Proceed in **small steps** in the **optimal direction** till a **stopping criterion** is met.
- **Gradient descent**: updates of the form: $x^{(t+1)} \leftarrow x^{(t)} - \eta_{(t)} \nabla f(x^{(t)})$

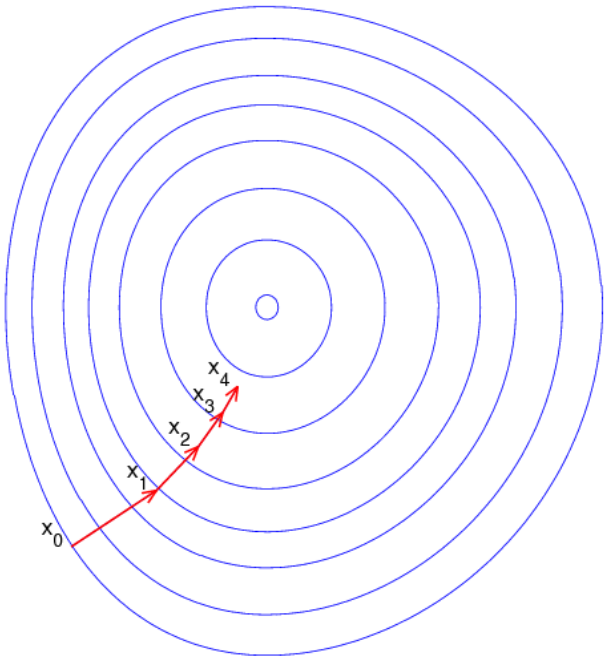

Figure: Illustration of gradient descent.The blue circles correspond to the function values at different points, while the red lines correspond to steps taken in

Convex functions

Pro: Guarantee of a global minima ✓

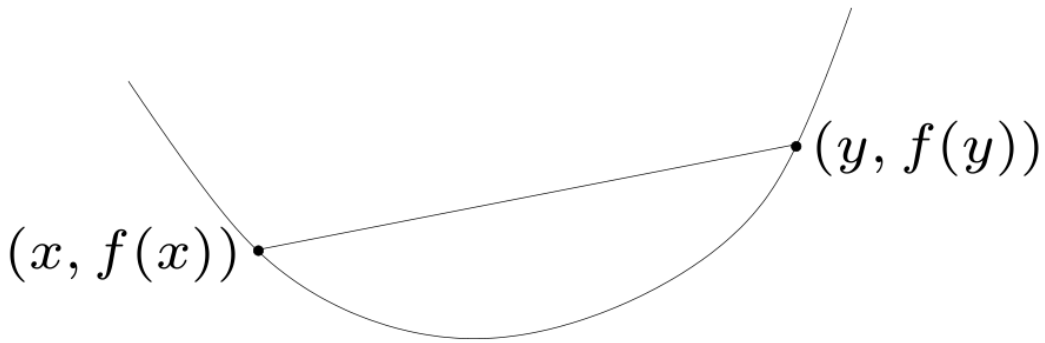

Figure: Illustration of a convex function. The line segment between any two points on the graph lies entirely above the curve.

Non-Convex functions

Pro: No guarantee of a global minima ✗

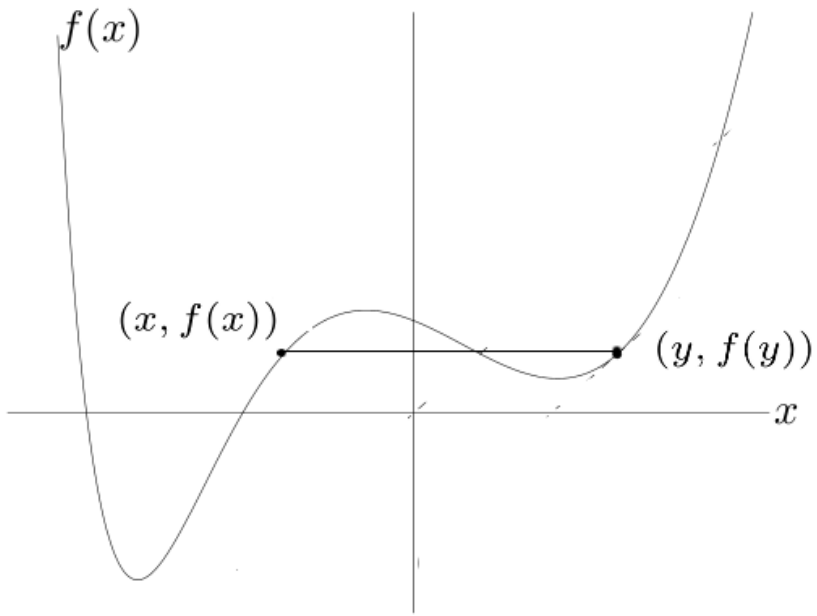

Figure: Illustration of a non-convex function. Note the line segment intersecting the curve.

Thank you!

Questions?

References I